A publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 4, No.3
November 2015

Thinking Skills PULLOUT

Azim Premji University

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COMMUNITY MATH CENTRE · RISHI VALLEY SCHOOL (KFI)

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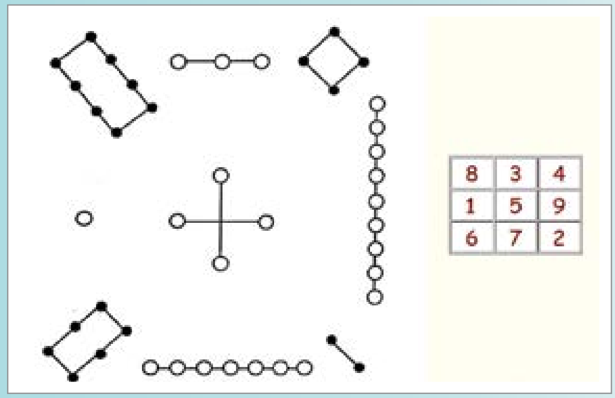
MAGIC SQUARES, A TOPIC WITH A RICH AND HOARY PAST

Magic squares have been a source of fascination for human beings for thousands of years. In the past, magic squares were essentially objects with supposedly magical properties, symbolising balance and harmony. In ancient China, the 3×3 magic square is said to form the basis of the practice of Feng Shui. Legend has it that an Emperor of old, swimming in the Yellow River, noticed a turtle with a pattern on its back; it is this pattern that gave rise to the Lo Shu magic Square: 12 + 8 + 5 + 9 = 34 = 1 + 11 + 16 + 6, etc
http://plaza.ufl.edu/ufkelley/magic/history.htm.

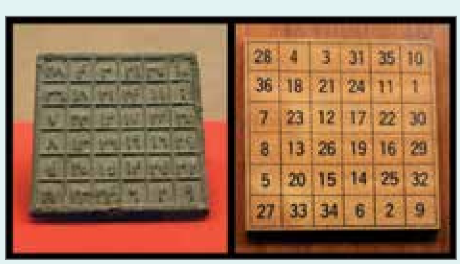


In India, the oldest known magic square is of the fourth-order; it is from Khajuraho, and it dates to the 10th century (see https://commons.wikimedia.org/wiki/File:2152085cab.png). This magic square has properties that go beyond the 'magic' requirement; its broken diagonals have the same total, equal to the magic sum of the square:

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Magic squares with this property are called *panmagic* squares. Indian mathematicians and Sanskrit grammarians of this period seem to have had strong leanings towards combinatorics and word-play, and this magic square was no doubt a result of that interest.

Here is a striking example of a sixth-order magic square from 12th or 13th century China, with the magic sum 111. It has been cast on an iron plate! Note the Arabic numerals; they point to a clear transmission of knowledge between mediaeval China and the Middle East.



"Yuan dynasty iron magic square" by BabelStone - Own work. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:Yuan_dynasty_iron_magic_square.jpg#/media/File:Yuan_dynasty_iron_magic_square.jpg

There is a great deal of mathematics that can be learnt using magic squares; for example, ideas of symmetry, ideas of modular arithmetic, ideas of repetitive patterns. One can with great ease bump into problems at the frontier of the subject. And one can have plenty of fun in the process too!

From the Editor's Desk . . .

Two themes dominate this issue of AtRiA: Archimedes and Magic Squares – an unlikely combination! Both are exceedingly rich topics to write about, with histories that go far back in time. Who cannot be both charmed and thrilled by the story of Archimedes? – the famous *Eureka* story, his computation of effective bounds for $\pi \left(3\frac{10}{71} < \pi < 3\frac{1}{7}\right)$ and his magical proofs for the formulas for volume of a cone, volume of a sphere, surface area of a sphere and area of a segment of a parabola. Here is a mathematician living in an age when algebra had yet to be invented, working with a numeration system that would have made even simple computations a challenge, and yet so utterly ahead of his time, dealing with complex ideas which would not become part of mainstream mathematics for close to two millennia; for example, ideas of limits and of the definite integral as the limit of a sum. And then there is the whole episode of his lost book surfacing in a library in Constantinople in the last decade of the 19th century. Just imagine: a book lying undiscovered for a thousand years! The whole story seems otherworldly. Dakshayani Suresh writes about the novel *The Sand Reckoner* in the 'Review' section, and Amrutha Manjunath writes in the 'Features' section about what has come to be known as "The Method" of Archimedes (to find area and volume).

Vinay Nair writes about "Magic Squares" in the 'Features' section; he considers the problem of how to construct third-order and fourth-order magic squares composed solely of prime numbers. Later in the issue, there is an article by Shailesh Shirali on "Theorems on Magic Squares," followed by an article where a fourth order magic square is constructed using Ramanujan's birth date (December 22, 1887) and the aforesaid theorems.

Also in the 'Features' section is a fascinating account by Gaurish Korpal of how a sheet of paper more than 1 km in length was folded twelve times – a feat considered to be impossible prior to this achievement! Following this, there is a continuation of the "3, 4, 5" article by Shailesh Shirali, in which some geometrical properties of a related triangle (with sides 4, 5, 6) are presented.

In the 'ClassRoom' section, Michael de Villiers studies an interesting geometrical problem which he had come across in a public examination, which offers scope for further analysis and for exploration using dynamic geometry software. A Ramachandran studies the relative placement of some well-known geometrical centres in an isosceles triangle. Tanuj Shah writes about the methodology of investigations in the mathematics classroom, and Sneha Titus and Swati Sircar continue their "Low Floor High Ceiling" (LFHC) series with an article on tangrams. CoMaC examines an intriguing problem that arises from a study of the constituents that go into the making of a dollar coin. Shailesh Shirali then rounds off the section with a continuation of the "How to prove it" series; the article studies the triangle-in-a-triangle problem. (The July 2015 issue of AtRiA had carried some material on the same problem.)

In 'TechSpace' Jonaki Ghosh writes about the "Hill Cipher;" this is the second part of the article on this topic. Hill ciphers form an important class of ciphers which can easily be studied at the school level; the student only needs to know about matrix operations and about modular arithmetic. Swati Sircar continues the article (begun in an earlier issue) on the generation of the conic sections using paper folding, and how one can simulate this construction using GeoGebra. Here she focuses on the ellipse and the hyperbola.

The 'Problem Section' has its usual mix of articles. In the 'Review' section, we have (in addition to the review of *The Sand Reckoner*) a review by Shashidhar Jagadeeshan of a well-known book by the great mathematician (and prolific writer of books and articles) Serge Lang. The 'Pullout' features a piece by Padmapriya Shirali on how one can develop higher order thinking skills in young children.

With this issue, AtRiA completes four years of publication. It is of interest to reflect on the experiences of these four years and of the challenges that lie ahead. Sourcing articles of general interest and in particular articles of interest to teachers of mathematics continues to be the greatest challenge which we face. We appeal to readers to contribute articles for publication. Scattered within the pages of each issue are also various "filler items" wherein we pose little challenges. It would be nice to hear from readers on these problems.

— Shailesh Shirali

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

This section has articles dealing with mathematical content, in pure and applied mathematics. The scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

In the ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The ClassRoom' is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

Contents contd.

TechSpace

'TechSpace' is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

Problem Corner

Reviews

Pullout

feature

The Man
The Methods of Archimedes
The Message

Amrutha Manjunath

Introduction

Elsewhere in this issue is a review of *The Sand Reckoner* by Gillian Bradshaw. That review and this article are dedicated to one of the most celebrated mathematicians in the world. Archimedes is perhaps most famous for the discovery of the Archimedes Principle and the invention of levers, pulleys, pumps, military innovations (like the siege engines) and the Archimedean Screw. His mathematical contributions include approximations of π and $\sqrt{3}$ accurate to several decimal places, proof of the quadrature of the parabola, formula for the area of a circle, and formulae of surface areas and volumes of several solid shapes. In this article, I have focused on two techniques (called Archimedes' Methods) by which he arrived at the formula of the volume of a sphere.

In October 1998, a French family in New York put a thousand-year-old manuscript up for public auction. This manuscript, which the family had acquired in the 1920s, turned out to be a lost Archimedean palimpsest. Byzantine monks in the 13th century had washed the original mathematical text and reused the parchment for Christian liturgical writings. In the early 20th century, Johan Heiberg had studied the same manuscript at Constantinople (present-day Istanbul) and

Keywords: *Archimedes, volume, cylinder, cone, sphere, method of exhaustion, equilibrium, lever*

identified it for the first time as work by Archimedes. It disappeared for several years during the aftermath of the Greco-Turkish War, and resurfaced in the possession of the French businessman whose descendants put it up for auction. From 1999 to 2008, the manuscript was subject to extensive imaging study and conservation at the Walters Art Museum in Baltimore in collaboration with scientists at Rochester Institute of Technology and Stanford University. Many Archimedean texts were recovered from this palimpsest, of which the work on Methods is especially interesting to many mathematicians.

The Method of Exhaustion

The Method of Exhaustion is a well-known technique using which the area of a figure can be found by visualizing it to be composed of constituent polygons that converge to the area of the containing shape. It is considered to be the ancient-Greek equivalent of the modern notion of limits. Among other results, Archimedes used the Method of Exhaustion to compute the volume of a sphere. I have discussed this method below using modern notation.

Consider the hemisphere in Figure 1. Archimedes imagined the hemisphere to be formed by the layering of cylinders inscribed within the solid. Let the radius of the hemisphere be r, and radii of each cylinder be $r_1, r_2, r_3, \ldots, r_n$. If there are n cylinders of equal height laid one on top of one another, it follows that the height of each cylinder is r/n. By the Pythagorean Theorem:

$$r_1 = r^2 - \frac{r^2}{n^2}, \quad r_2 = r^2 - \frac{(2r)^2}{n^2}, \quad \ldots,$$
$$r_{n-1} = r^2 - \frac{(n-1)r^2}{n^2}, \quad r_n = r^2 - \frac{(nr)^2}{n^2}.$$

As the number of cylinders increases, and the height of each cylinder correspondingly decreases, the sum of volumes of the cylinders is a closer and closer approximation to the volume of the hemisphere. Therefore, as n approaches infinity, the sum of the volumes of the cylinders equals the volume V of the hemisphere. That is,

$$\begin{aligned} V &= \lim_{n\to\infty}\left(\pi r_1^2\,\frac{r}{n} + \pi r_2^2\,\frac{r}{n} + \pi r_3^2\,\frac{r}{n} + \cdots + \pi r_n^2\,\frac{r}{n}\right) \\ &= \lim_{n\to\infty} \pi\frac{r}{n}\left(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2\right). \end{aligned}$$

Substitute $r_j = r^2 - (jr/n)^2$ for $1 \le j \le n$:

$$\begin{aligned} V &= \lim_{n\to\infty} \pi\frac{r^3}{n}\left(n - \frac{(1^2 + 2^2 + 3^2 + \cdots + n^2)}{n^2}\right) \\ &= \pi r^3 - \pi r^3 \lim_{n\to\infty}\frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right). \end{aligned}$$

Now use the formula $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$:

$$\begin{aligned} V &= \pi r^3 - \pi r^3 \lim_{n\to\infty}\left(\frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right) \\ &= \pi r^3 - \pi r^3 \lim_{n\to\infty}\left(\frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right) \\ &= \pi r^3 - \frac{\pi r^3}{3} = \frac{2\pi r^3}{3}. \end{aligned}$$

Since V is half the required volume, the volume of the sphere with radius r is given by $\frac{4}{3}\pi r^3$.

A similar argument can be made to obtain the same result if the hemisphere is thought to be circumscribed by a layering of cylinders; see Figure 2. The solid shape is in fact "sandwiched" between the inscribed and circumscribed cylinders. As n tends to infinity, the two stacks of cylinders converge to the form of the hemisphere.

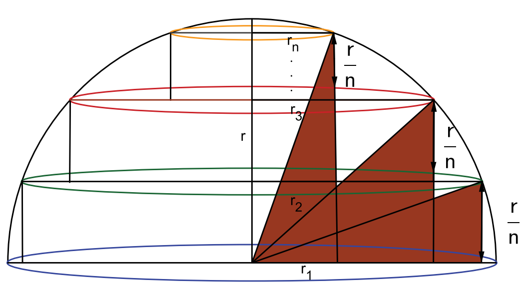

Figure 1.

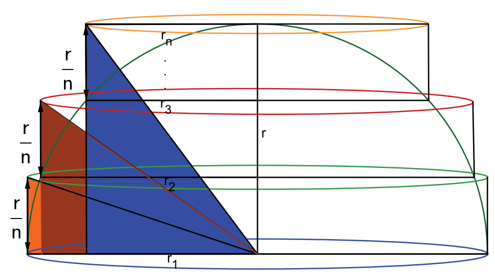

Figure 2.

The Method of Equilibrium

Now I will discuss the second technique by which Archimedes arrived at the same result for the volume of a sphere. This technique, known as the Method of Equilibrium, was found in the lost palimpsest. It sheds light on a very uniquely Archimedean way of thinking about surface areas and volumes of solid shapes, and employs an argument that resonates with the modern notion of integral calculus.

There has been considerable debate among mathematicians about which Method of Archimedes is the superior one. Archimedes conceptualized notions of limits and integration well before calculus emerged as a powerful mathematical tool, so both methods contain ideas much ahead of their time. Historians of mathematics such as Howard Eves argue that the Method of Exhaustion is "sterile" because its elegance is apparent only if the result is already known. While this is debatable, the Method of Equilibrium is unique for Archimedes' use of mechanics to prove a purely mathematical result. Archimedes himself is said to have preferred the Method of Exhaustion because he felt that it was mathematically more rigorous. Perhaps this was born out of his innate preference for pure mathematics to mechanical inventions. However, in the words of E.T. Bell, "To a modern all is fair in love, war, and mathematics." Maybe the equilibrium argument is considered more elegant today because there is something enchanting when borders between related disciplines melt to reveal how closely interlinked the disciplines really are.

To find the volume of a sphere by the Method of Equilibrium, it helps to think of the solid as cut up into a large number of very thin strips hung end to end on an imaginary lever. This proof compares the moments of two solids when placed on the lever. Since volume is proportional to mass, moment of the solid can be defined as the product of its volume and lever length (the distance from the point about which the shapes are hung to the centroid of the volume).

Figure 3 is a cross-sectional view along the equator of the sphere. Here $AO = OB = 2r$. Consider the cylinder and cone of revolution obtained by rotating rectangle $OPSB$ and triangle OCB about the AB axis. Suppose thin vertical slices of thickness Δx are cut from the three solids at distance x from O. The approximate volumes of the sections of each solid are deduced to be:

Sphere: The equation of the circular cross-section of the sphere is $(x-r)^2 + y^2 = r^2$, i.e., $y^2 = x(2r - x)$. Therefore the volume of revolution of the slice of sphere with thickness Δx and height y is $\pi y^2 \Delta x = \pi x(2r - x)\Delta x$.

Cone: The volume of revolution of the slice of cone with thickness Δx and height x is $\pi x^2 \Delta x$.

Cylinder: The volume of revolution of the slice of cylinder with thickness Δx and height $2r$ is $\pi r^2 \Delta x$.

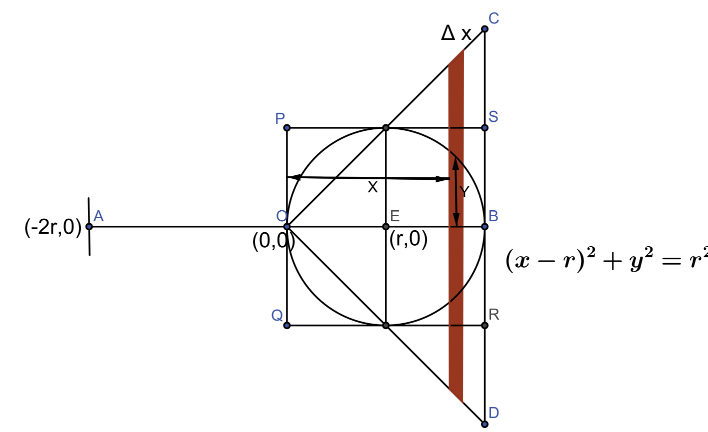

Figure 3.

If the slices from the sphere and the cone are imagined to be stacked at A, they form a single point mass. Their combined moment about the point O is given by:

$$\text{Sum of volumes of slices of sphere and cone} \\ \times \text{length } OA = (\pi x(2r - x)\Delta x + \pi x^2 \Delta x)2r \\ = 4\pi r^2 x \Delta x.$$

The moment about O of the slice cut from the cylinder (when its position is unchanged) is given by:

$$\text{Volume of slice of cylinder} \\ \times \text{distance from } O \text{ to slice} = (\pi r^2 \Delta x) * (x) \\ = \pi r^2 x \Delta x.$$

Therefore, the moment about O of the slices of the cone and sphere is 4 times the moment about O of the slice of the cylinder. When a large number of such slices are added together, the following expression is obtained:

$$2r \times \text{volume of sphere} + \text{volume of cone} \\ = 4r \times \text{volume of cylinder}.$$

Here, $4r$ is the length of the lever arm. Since the volume of the cone is known to be $\frac{\pi(2r)^3}{3}$ and that of the cylinder to be $(2r)\pi r^2 = 2\pi r^3$, we get:

$$2r \times \text{volume of sphere} + \frac{8\pi r^3}{3} = 8\pi r^4.$$

Therefore, the volume of the sphere is $\frac{4\pi r^3}{3}$.

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feature

3, 4, 5 ... And other memorable triples — Part II

SHAILESH SHIRALI

In Part I of this article we had showcased the triple (3, 4, 5) by highlighting some of its properties and some configurations where it occurred naturally. We now attempt to extend this to other triples of consecutive integers. To begin with, we study the two 'siblings' of (3, 4, 5), namely, the triples (2, 3, 4) and (4, 5, 6). We start first with the elder sibling, (4, 5, 6). (We do need to show the older ones some respect, don't we?)

The triple 4, 5, 6

In Figure 1 we see a sketch of a triangle ABC with sides 4, 5, 6 (with $a = 6$, $b = 5$, $c = 4$). Is there anything special about the triangle? Let's do some exploration using *GeoGebra*.

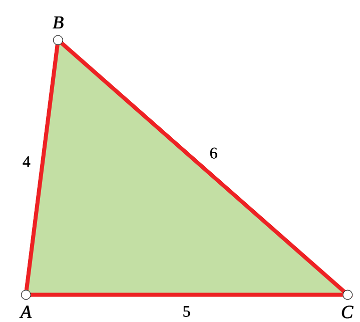


Figure 1.

Keywords: Triangle, consecutive integers, triple, double angle, sine rule, cosine rule, Pythagoras

Figure 1 shows a *GeoGebra* sketch of the triangle. We start by measuring the angles of the triangle (using the tool available in *GeoGebra*). Here is the output:

$$\measuredangle A = 82.82^\circ, \qquad \measuredangle B = 55.77^\circ, \qquad \measuredangle C = 41.41^\circ.$$

Examining the data, we quickly notice that 82.82 is twice 41.41, in other words: $\measuredangle A = 2\measuredangle C$. Right away we have uncovered something notable and of interest!

But wait: this relation has been *numerically determined*. Could it be the case that if we compute both angle measures to more decimal places than shown above, the above relation will turn out to be only approximate and not exact? How can we check whether or not $\measuredangle A$ is *exactly* twice $\measuredangle C$?

We can do so using trigonometry. Let us compute the cosines of all three angles of the triangle using the cosine rule:

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{25+16-36}{2\times 20} = \frac{1}{8},$$
$$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{16+36-25}{2\times 24} = \frac{9}{16},$$
$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{36+25-16}{2\times 30} = \frac{3}{4}.$$

To see if $\measuredangle A = 2\measuredangle C$ as suggested by the empirical evidence, we must check whether $\cos A = 2\cos^2 C - 1$ (for we have the identity $\cos 2\theta = 2\cos^2\theta - 1$ which is true for any angle θ). We have:

$$2\cos^2 C - 1 = 2\left(\frac{3}{4}\right)^2 - 1 = \frac{9}{8} - 1 = \frac{1}{8} = \cos A,$$

and since both $\measuredangle A$ and $\measuredangle C$ are acute angles, the verification is complete. *So the relation* $\measuredangle A = 2\measuredangle C$ *is indeed exact.*

The same property can be proved by a geometric argument which may be preferred by some. In Figure 2 (a) we have redrawn the 4-5-6 triangle with the perpendicular BD from vertex B to side AC. Our first task is to find the length x of AD. We shall make use of the Pythagorean theorem to do so. Let h be the length of BD. Then we have:

$$h^2 + x^2 = 4^2,$$
$$h^2 + (5-x)^2 = 6^2,$$

hence by subtraction: $(5-x)^2 - x^2 = 6^2 - 4^2$, i.e., $25 - 10x = 20$. This yields $x = 1/2$.

Let E be the point on side AC such that $AE = 1$ unit; see Figure 2 (b). Join BE. Since $DE = DA$, it follows that $BE = BA$. Also $EC = 5 - 1 = 4$ units. So we have $AB = BE = EC$. Hence $\measuredangle BEA = 2\measuredangle BCA$, and also $\measuredangle BEA = \measuredangle BAE$. It follows that $\measuredangle BAC = 2\measuredangle BCA$, i.e., $\measuredangle A = 2\measuredangle C$.

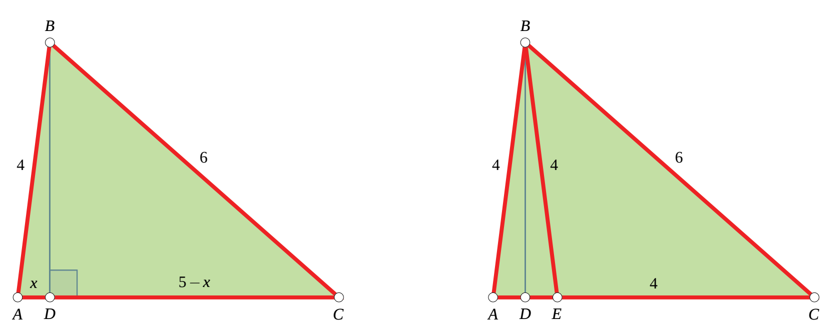
A Stronger Property

We now prove something much more striking:

Theorem 1. *There is only one triple of consecutive integers with the property that the triangle with these numbers as its side lengths has one angle which is twice another one. This is the triple* $(4, 5, 6)$.

Let the sides of the triangle be $n, n+1, n+2$. Let the triangle be labelled ABC so that $a = n+2$, $b = n+1$, $c = n$. Since $a > b > c$, we have $\measuredangle A > \measuredangle B > \measuredangle C$. So if one angle of the triangle is twice another, one of the following must be true: (i) $\measuredangle A = 2\measuredangle B$ (ii) $\measuredangle B = 2\measuredangle C$ (iii) $\measuredangle A = 2\measuredangle C$.

There are now two ways of proceeding. One is to use the cosine rule. This works, but the algebra is messy. The other, which is more interesting as well as more efficient, and which we prefer, is to use a geometric Pythagoras-style theorem which is striking by itself.



(a) (b)

Figure 2.

Theorem 2. *Let* $\triangle ABC$ *have sides* a, b, c. *Then the relation* $\measuredangle A = 2\measuredangle B$ *is true if and only if* $a^2 = b(b+c)$.

Proof of Theorem 2: Forward implication. We first tackle the statement: if $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$. (This is the 'only if' part of the theorem.) We offer a trigonometric proof of the result. Let $\measuredangle B = \theta$; then $\measuredangle A = 2\theta$ and $\measuredangle C = 180^\circ - 3\theta$. Hence we have $\sin A = \sin 2\theta$ and $\sin C = \sin 3\theta$. The sine rule yields:

$$\frac{a}{\sin 2\theta} = \frac{b}{\sin\theta} = \frac{c}{\sin 3\theta}.$$

From the first equality we get:

$$a = b\cdot\frac{\sin 2\theta}{\sin\theta} = 2b\cos\theta, \qquad \therefore\ \cos\theta = \frac{a}{2b}.$$

The second equality yields:

$$c = b\cdot\frac{\sin 3\theta}{\sin\theta} = b\cdot\frac{3\sin\theta - 4\sin^3\theta}{\sin\theta}$$
$$= b\left(3 - 4\sin^2\theta\right)$$
$$= b\left(4\cos^2\theta - 1\right).$$

Substituting for $\cos\theta$ in this relation, we get:

$$c = b\left(\frac{a^2}{b^2} - 1\right) = \frac{a^2 - b^2}{b},$$
$$\therefore\ a^2 = b^2 + bc = b(b+c),$$

as claimed.

Proof of Theorem 2: Reverse implication. Now we tackle the 'if' part of the theorem, namely: if $a^2 = b(b+c)$, then $\measuredangle A = 2\measuredangle B$. Once again, we offer a trigonometric proof of the result. We use the sine rule together with the following beautiful identity whose proof we leave as an exercise:

$$\sin^2 A - \sin^2 B = \sin(A+B)\ \sin(A-B).$$

The sine rule tells us that for any triangle ABC, we have $a/\sin A = b/\sin B = c/\sin C =$ some constant k. (In fact, k is the circum-diameter of the triangle, i.e., it is twice the radius of the circumcircle. But we do not need this information right now.)

From the relation $a^2 = b(b+c)$ we get $a^2 - b^2 = bc$, which tells us that $a > b$ and therefore that $\measuredangle A > \measuredangle B$. The same relation also yields, by the sine rule:

$$\sin^2 A - \sin^2 B = \sin B\ \sin C.$$

Using the trigonometric identity quoted above, we get:

$$\sin(A+B)\ \sin(A-B) = \sin B\ \sin C.$$

Since $A + B + C = 180^\circ$, we have $\sin(A+B) = \sin C$. Since $\sin C \neq 0$, we get:

$$\sin(A-B) = \sin B.$$

Since $A - B$ and B lie between 0° and 180° and have equal sine, they are either equal angles or they are supplementary angles. The latter possibility leads to $(A-B) + B = 180^\circ$, i.e., $A = 180^\circ$, which is absurd. Hence this case does not hold. It follows that $A - B = B$, i.e., $\measuredangle A = 2\measuredangle B$.

There is also an elegant geometric proof of the result (both parts: forward implication as well as reverse implication), which we shall discuss later.

Proof of Theorem 1. We now use Theorem 2 to prove Theorem 1. We consider the three possibilities in turn.

Case (i): If $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$, hence:

$$(n+2)^2 = (n+1)(2n+1),$$
$$\therefore\ n^2 + 4n + 4 = 2n^2 + 3n + 1,$$
$$\therefore\ n^2 - n - 3 = 0.$$

This equation has roots $n = \frac{1}{2}(1 \pm \sqrt{13})$. These are not positive integers (or even rational numbers), so we do not get any solution from this possibility.

Case (ii): If $\measuredangle B = 2\measuredangle C$, then $b^2 = c(c+a)$, hence:

$$(n+1)^2 = n(2n+2),$$
$$\therefore\ (n-1)(n+1) = 0.$$

This yields $n = \pm 1$. Only the positive sign is of interest to us. However, the triangle corresponding to $n = 1$ has sides 1, 2, 3 and so is degenerate: it is 'flat', with angles 180°, 0° and 0°. Note that the solution is not 'wrong'. For, this triangle has $\measuredangle B = 0^\circ = \measuredangle C$, which means that we do have the relation $\measuredangle B = 2\measuredangle C$! But it is of no interest to us, so we move on.

Case (iii): If $\measuredangle A = 2\measuredangle C$, then $a^2 = c(c+b)$, hence:

$$(n+2)^2 = n(2n+1),$$
$$\therefore\ n^2+4n+4 = 2n^2+n,$$
$$\therefore\ n^2-3n-4 = 0,$$
$$\therefore\ (n+1)(n-4) = 0.$$

The last equation has roots $n = -1$ and $n = 4$. We finally do get a positive integral root, $n = 4$, and this yields a genuine, well-behaved triangle: a triangle with sides $4, 5, 6$. This yields a solution to the stated problem.

It follows that there is precisely one triangle with the stated property: the one that has sides $4, 5, 6$.

In closing we may say that the triple $(4, 5, 6)$ can lay its own claim to fame, with its own pleasing property, just like its better known sibling $(3, 4, 5)$.

A Geometric Proof of Theorem 2

Some readers may prefer to see a *geometric* proof of Theorem 2 (we had earlier given a proof using trigonometry). We offer one such proof here.

First we deal with the forward implication: if $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$. The relevant configuration is shown in Figure 3.

We need an auxiliary construction. Draw a circle tangent to side BC at B and passing through vertex A. (The circle may be constructed as follows: draw a perpendicular to BC through B, and draw the perpendicular bisector of side AB; the point where these two lines meet is then the centre of the desired circle. These auxiliary construction lines have not been shown in Figure 3, to avoid a visual clutter.)

Extend side CA beyond vertex A to meet the circle again at point D. Draw segments BD and AD, as shown. Let AD have length d. Let $\measuredangle ABC = \theta$; then $\measuredangle BAC = 2\theta$ as per the given data.

From the fact that CB is tangent to the circle at B, two deductions follow: (i) $\measuredangle ABC = \measuredangle ADB$, i.e., $\measuredangle ADB = \theta$; this follows from the "angle in the alternate segment" theorem; (ii) $CB^2 = CA \times CD$, i.e., $a^2 = b(b+d)$; this is true because CAD is a secant.

Since $\measuredangle BAC = \measuredangle ADB + \measuredangle ABD$, it follows that $\measuredangle ADB = \theta$. Hence $\triangle ADB$ is isosceles, with $AD = AB$. So $d = c$. Combining this with deduction (ii), above, we see that $a^2 = b(b+c)$, as claimed.

Now for the reverse implication: if $a^2 = b(b+c)$, then $\measuredangle A = 2\measuredangle B$. We use the same figure for the proof, with the same auxiliary construction. The configuration is depicted in Figure 4. As earlier, we have drawn a circle tangent to side BC at B and passing through vertex A; then we have extended side CA beyond vertex A to meet the circle again at point D, and drawn segments BD and AD. Let AD have length d.

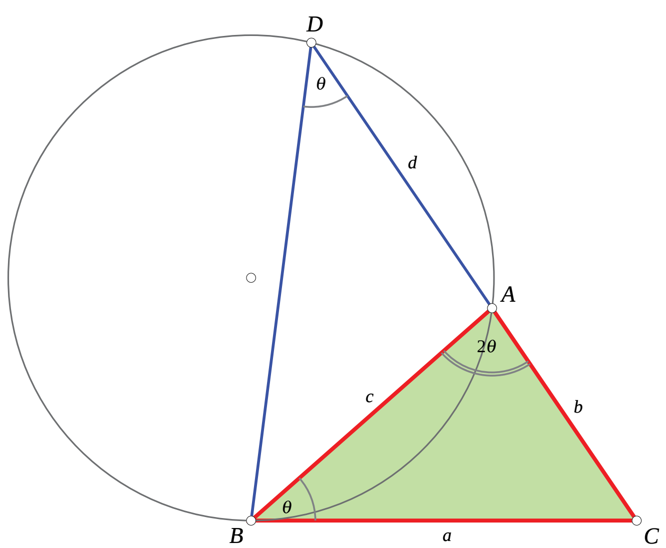


Figure 3. Given that $\measuredangle A = 2\measuredangle B$, to show that $a^2 = b(b+c)$

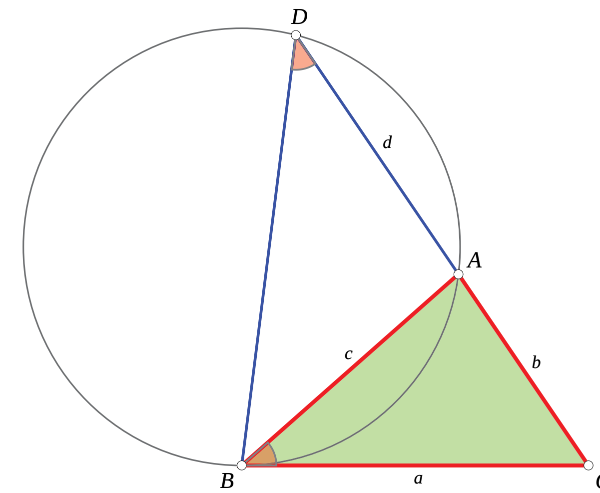


Figure 4. Given that $a^2 = b(b+c)$, to show that $\measuredangle A = 2\measuredangle B$

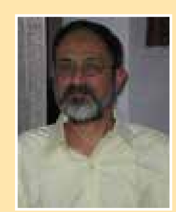
Since CB is tangent to the circle at B, and CAD is a secant, we have the following relation: $CB^2 = CA \times CD$, i.e., $a^2 = b(b+d)$.

But we also have the given relation $a^2 = b(b+c)$. Comparing the two relations, we conclude that $c = d$, i.e., $AB = AD$. Hence $\measuredangle ABD = \measuredangle ADB$. And since $\measuredangle BAC = \measuredangle ABD + \measuredangle ADB$, it follows that $\measuredangle BAC = 2\measuredangle ADB$.

But we also have $\measuredangle ABC = \measuredangle ADB$, by the "angle in the alternate segment" theorem. Hence $\measuredangle BAC = 2\measuredangle ABC$, i.e., $\measuredangle A = 2\measuredangle B$, as claimed.

Appendix: Integer triples associated with this theorem

Associated with the Pythagorean theorem we have the number theoretic problem of generating Pythagorean triples. In the same way, associated with the main result derived in this article, we have another interesting number theoretic problem: that of generating integer triples (a, b, c) which satisfy the equation $a^2 = b(b+c)$. We may want to impose the additional condition that a, b, c are coprime, just as we did in the case of Pythagorean triples. We already have one example of such a triple: $(6, 4, 5)$. Are there any others? Yes; and they are quite easy to find. We leave this question for the reader to tackle: that of finding an efficient and effective algorithm for generating all coprime integer triples (a, b, c) which satisfy the equation $a^2 = b(b+c)$. We will take up a study of this equation in a subsequent article.



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feature

Algebracadabra

Magic Squares . . .

. . . with prime entries

Magic squares have been a subject of fascination for centuries. Probably it is the elegance and the simplicity in the subject that attract people. Here we explore the question of how to construct magic squares composed solely of prime numbers.

VINAY NAIR

A magic square is an array of numbers in an $n \times n$ square grid arranged in such a way that the sum of the numbers in each row, each column and each of the two main diagonals is the same number (called the 'Magic Sum' of that square).

There are various ways of classifying magic squares; one is by the number of rows and columns. Thus we have *Odd-order Magic Squares,* with an odd number of rows, and *Even-order Magic Squares*, with an even number of rows. Even-order Magic Squares can be further classified into *Singly-even Magic Squares*, for which the number of rows is even but not a multiple of 4, and *Doubly-even Magic Squares*, for which the number of rows is a multiple of 4.

Constructing 3 × 3 Magic Squares

The numbers in any 3 × 3 magic square will always be formed using three Arithmetic Progressions (APs). Below is a 3 × 3 magic square where three APs have been placed in a particular pattern.

42	4	32
16	26	36
20	48	10

Keywords: *Magic square, entry, prime, arithmetic progression*

Observe the following:

- The square is made up of three APs: (4, 10, 16), (20, 26, 32) and (36, 42, 48).
- Each row of the square contains one term of each of the APs. So also each column.
- The three APs have the same common difference (in this case 6).
- The first terms of the APs themselves form an AP, namely: (4, 20, 36).

To construct a 3 × 3 magic square, one has only to remember the above rules. This gives us the liberty to create 3 × 3 magic squares with arbitrary numbers of our choice provided they form APs related as above.

A thought occurs to us at this stage: *Can we construct magic squares composed of only prime numbers?*

Constructing 3 × 3 magic squares using only prime numbers

Constructing a 3 × 3 magic square using only prime numbers (we shall refer to such a square as a *Prime Magic Square*) is possible but challenging because we first need to list sufficiently many APs composed of primes, and they have to be interconnected in the right way. Shown below is an example of a prime magic square.

101	29	83
53	71	89
59	113	41

In going about this task, the following observation may be kept in mind: *If three prime numbers exceeding* 3 *form an AP, then the common difference of the AP is necessarily a multiple of* 6. The reader is invited to supply a proof of the statement. Using this observation simplifies the search for prime APs.

But how often do we come across prime APs with four or more terms? This is where constructing a 4 × 4 prime magic square becomes challenging, because four-term prime APs are a less frequent sight.

Constructing a 4 × 4 magic square

There are various methods for constructing 4×4 magic squares. Shown below is one of the methods used for positioning the numbers within it. The APs used here are:

AP(#1) = (1, 2, 3, 4), AP(#2) = (5 ,6, 7, 8),

AP(#3) = (9, 10, 11, 12), AP(#4) = (13, 14, 15, 16).

Here is how we arrange the different APs. We start with AP #1:

	1		
			2
		3	
4			

We may assign a symbol to fix or recall the above pattern: 2-4-3-1. Thus, the first term of the AP is placed in cell #2 of the first row, the second term of the AP is placed in cell #4 of the second row, the third term of the AP is placed in cell #3 of the third row, and the fourth term of the AP is placed in cell #1 of the fourth row.

Note the appearance of the L-shaped movement which reminds us of the knight move in chess. This is a theme which often crops up in the construction of magic squares.

Next we arrange the terms of AP #2:

	1		7
	8		2
5		3	
4		6	

This arrangement may be denoted by the symbol 4-2-1-3.

Following this step, we go on to arrange the terms of AP #3:

	1	12	7
11	8		2
5	10	3	
4		6	9

This arrangement may be denoted by the symbol 3-1-2-4. Here the term 9 (the first term in AP #3) is placed in the cell diagonally alternate to 8. (By 'diagonally alternate' we mean the cell which is 2 units to the right and 2 units below the starting cell. That is, the cell diagonally alternate to (i, j) is $(i+2, j+2)$, where addition is done modulo 4.) The second term in the AP is 10 which is placed in the cell diagonally alternate to 7, and it goes in this pattern for the remaining terms and for the final AP, the pattern being 1-3-4-2:

14	1	12	7
11	8	13	2
5	10	3	16
4	15	6	9

The final array is a magic square of order 4, with magic sum 34. The positions of numbers 1 to 16 define one way of positioning the numbers in the grid to get a magic square. (There are of course many other ways of doing this.) We can now take any four four-term APs and place them in these positions to get a 4 × 4 magic square.

Constructing a 4 × 4 prime magic square

Using the following selection of four-term APs composed of prime numbers,

AP(#1) = (11, 17, 23, 29),
AP(#2) = (41, 47, 53, 59),

AP(#3) = (61, 67, 73, 79),
AP(#4) = (251, 257, 263, 269),

we obtain the following 4 × 4 magic square with a magic sum of 400:

257	11	79	53
73	59	251	17
41	67	23	269
29	263	47	61

By selecting different sets of primes, we may generate infinitely more such magic squares. Try it out on your own!

Above we have a beautiful example of a 4 × 4 Prime Magic Square. But the square has an interesting feature which makes it different from the earlier 3 × 3 square: the first terms in the four APs (11, 41, 61, 251) do not themselves form an AP, yet the APs result in a Magic Square. This makes matters simpler for us: if we wish to construct a 4 × 4 Prime Magic Square, we only need four prime APs with the same common difference; their first terms need not form an AP. (In passing, we remark that there are various other combinations for constructing a 4 × 4 magic square where one starts from other cells and takes a knight move in a different direction. There are also methods of constructing such squares without taking the knight move.)

Constructing a 4 × 4 prime magic square without APs

Is it possible to construct a Prime Magic Square without using prime APs? Well, we will never know until we try, will we? Here's what we find when we make the attempt. We use the following four sequences of primes (note that they do not form APs but possess a definite structure):

Sequence(#1) = (29, 31, 59, 61),
Sequence(#2) = (71, 73, 101, 103),
Sequence(#3) = (149, 151, 179, 181),
Sequence(#4) = (197, 199, 227, 229).

Here we have two pairs of twin primes in every sequence such that the differences between the second and third terms in all the four sequences are the same. (Note: Twin primes are a pair of prime numbers that differ by 2. So they are a pair of consecutive odd numbers, both of which are prime.) Once again, the four sequences have to be similar in nature with regard to the common difference, but the first terms of the four sequences need not form an AP.

Using these we form the following 4 × 4 Prime Magic Square. The numbers in the four sequences are inserted into the cells according the same pattern used earlier: 2-4-3-1 for Sequence #1, 4-2-1-3 for Sequence #2, and so on. Here is what we get when we do this for all the four sequences:

199	29	181	101
179	103	197	31
71	151	59	229
61	227	73	149

This line of thinking opens up the possibility of exploring more types of 4 × 4 Prime Magic Squares, because the restriction of APs has been removed, as also the restriction that the first terms of the sequences should be in an AP/ Sequence. A layman would be happy going this far. But for the mathematically inclined person, there arises the following question: *What is the logic behind Magic Squares?* Here we encounter the fascinating algebra of Magic Squares.

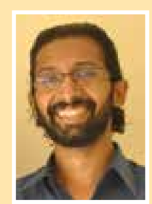
We select the four sequences according to the following pattern (note their structure):

Sequence(#1) = $(a, a+x, a+y, a+z)$,
Sequence(#2) = $(b, b+x, b+y, b+z)$,
Sequence(#3) = $(c, c+x, c+y, c+z)$,
Sequence(#4) = $(d, d+x, d+y, d+z)$.

Any four such sequences can be placed in the grid according to the pattern described earlier, and it will result in a Magic Square whose magic sum is $a+b+c+d+x+y+z$. Here is the result:

$d+x$	a	$c+z$	$b+y$
$c+y$	$b+z$	d	$a+x$
b	$c+x$	$a+y$	$d+z$
$a+z$	$d+y$	$b+x$	c

Closing remarks. The topic of Magic Squares can be further explored. What has been seen till now is just a tip of the iceberg. It is a good topic for exploration not just because of its recreational aspect but also because there are many areas of application. The subject indeed has Magic in it, for it manages to attract both young and old, student and teacher, layman and mathematician.



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feature

Say Crease!

Folding Paper in Half

Miles Please

When I was a small child I read the following claim about paper folding: "It is impossible to fold any piece of paper in half more than eight times no matter how big, small, thin or thick the paper is" ([1], pp. 32-43). At that time I accepted the fact after some experimentation. But last month, to my amazement, I discovered that a grade 11 student in California, Britney C. Gallivan, had mathematically disproved the above statement in 2002 [1]. This article discusses the proof given by Gallivan regarding "folding paper in half, n times".

Gaurish Korpal

Introducing the Problem: Gallivan's Rules

We first state the rules to be followed while folding a sheet of paper in half, as enunciated by Britney Gallivan:

1. A single rectangular sheet of paper of any size and uniform thickness can be used.
2. The fold has to be in the same direction each time. (Hence the fold lines are all parallel to each other.)
3. The folding process must not tear the paper. (That is, it must not introduce any discontinuities.)
4. When folded in half, the portions of the inner layers which face one another must almost touch one another.
5. The average thickness or structure of material of paper must remain unaffected by the folding process.
6. A fold is considered complete if portions of all layers lie in one straight line (called *folded section*).

Keywords: Gallivan, paper folding, half fold, geometric progression, creep section, radius section, series summation, quadratic equation

Following the rules, we claim that: *The length of the given sheet of paper decides the number of times we can fold it in half.* Thus if we have a sheet of paper of given length, we can calculate the number of times we can fold it theoretically (allowing a reasonable amount of manpower and time).

HOW TO REACH THE SUN ...ON A PIECE OF PAPER

A poem by Wes Magee

Take a sheet of paper and fold it,
and fold it again,
and again, and again.
By the 6th fold it will be 1-centimeter thick.
By the 11th fold it will be
32-centimeter thick,
and by the 15th fold - 5-meters.
At the 20th fold it measures 160-meters.
At the 24th fold - 2.5-kilometers,
and by fold 30 it is 160-kilometers high.
At the 35th fold it is 5000-kilometers.
At the 43rd fold it will reach the moon.
And by the fold 52
will stretch from here
to the sun!
Take a piece of paper.
Go on.
TRY IT!

Absolute folding limit

Geometric series. There is a short poem by Wes Magee titled "*How to reach the sun ...on a piece of paper*" ([2], page 19), which illustrates the geometric series involved in folding paper in half. After 52 folds (if possible), the width of the folded paper will be approximately equal to the distance between the sun and the earth!

Every time we fold the paper in half, we double the number of layers involved. We have to fold 2^{n-1} sheets of paper for the n^{th} fold. Thus for each successive fold we need more and more energy. Initially this was thought to be the reason for our inability to fold a piece of paper more than 8 times. But, as stated earlier, the strength of the arm is not the limiting factor for the number of folds.

Understanding folds. After each fold, some part of the middle section of the previous layer becomes a rounded edge. The radius of the rounded portion is one half of the total thickness of folded paper; see Figure 1.

Initially the radius is small as compared with the length of the remaining part of sheet. As the folds

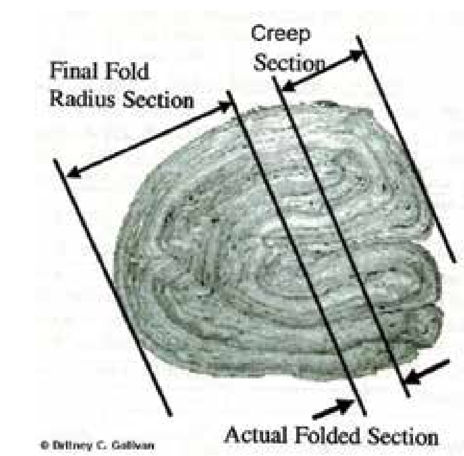

Figure 1. Paper folded in half 12 times illustrating the decrease in folded section and increase in radius section caused due to continued folding which leads to fold losses [©Britney C. Gallivan]; source: [1], [4]

begin nearing their final thickness, the curved portion becomes more prominent and begins taking up a greater percentage of the volume of the paper. The radius section is the part of the paper 'wasted' in connecting the layers.

The section that projects past the folded section on the side opposite the radius section is called the *creep* (Figure 1). It is caused by the difference in lengths of layers due to the rounded section of the fold layers having different radii and circumferences.

The limit to the number of folds is reached when a fold has been completed but there is not enough volume or length in the folded section of the paper to fill the entire volume needed for the radius section of the next fold. Thus while making folds there is loss of paper in the form of *radius section* and *creep section*.

Limit formula. Since the *radius* and *creep* sections are semicircular, the length to height ratio of the paper being folded has to be greater than π to allow one more successful fold to occur. *If a folded section's length is less than π times the height, the next fold cannot be completed.*

Let t be the thickness of a sheet of paper. On the first fold, we lose a semicircle of radius t, so the length lost is πt ('lost' in the sense of 'not contributing to the length'). On the second fold, we lose a semicircle of radius t and another semicircle of radius $2t$, so the length lost is

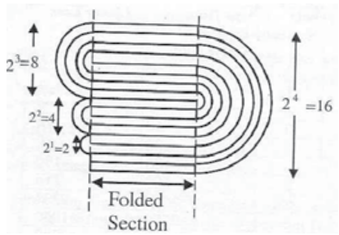

Figure 2. The folded portion and the circular portions [©Britney C. Gallivan]; source: [1], [4]

$\pi t + 2\pi t$ (see Figure 2). Similarly, on the third fold, the length lost is $\pi t + 2\pi t + 3\pi t + 4\pi t$. Generalising this argument, on the n^{th} fold, the length of paper lost is $\pi t + 2\pi t + 3\pi t + \cdots + 2^{n-1}\pi t$.

If n folds are required, the total length lost (L) is:

$$\begin{aligned} L = \pi t + (\pi t + 2\pi t) + (\pi t + 2\pi t + 3\pi t + 4\pi t) \\ + \cdots + (\pi t + 2\pi t + 3\pi t + \cdots + 2^{n-1}\pi t). \quad (1) \end{aligned}$$

Thus the cumulative losses or *minimum length of paper required*, L, of each and every layer and fold can be described by the following equation:

$$\begin{aligned} L &= \sum_{i=1}^{n} \sum_{k=1}^{2^{i-1}} \left(2^{i-1}\pi t - (k-1)\pi t\right) \\ &= \pi t \sum_{i=1}^{n} \sum_{k=1}^{2^{i-1}} \left(2^{i-1} - (k-1)\right). \qquad (2) \end{aligned}$$

The first summation, from $i = 1$ to $i = n$, is the summation that increments each physical fold. The second summation, from $k = 1$ to $k = 2^{i-1}$, is a term to allow for the amount of loss of each sheet in a particular fold. Here, the upper limit of 2^{i-1} is one half of the thickness of that fold, or the radius of the outer folded layer of the fold. Since the second summation must end at $2^{i-1}\pi t$ (corresponding to the i^{th} fold), we introduce another variable, k, to insert $(2^{i-1} - 1)$ steps of size πt before the final step. The $(k-1)$ term steps between the thickness of each sheet in a fold by the sheet's thickness, t. Thus, the above formula computes the radius of each fold and then the length of each sheet in the semicircular fold.

In equations (1) and (2), we calculate the length lost in the i^{th} fold by doing a summation in a "climbing a ladder" fashion and "descending a ladder" fashion (by using the variable k), respectively.

We can simplify the above double summation formula as follows:

$$\begin{aligned} L &= \pi t \sum_{i=1}^{n} \left(\sum_{k=1}^{2^{i-1}} 2^{i-1} - \sum_{k=1}^{2^{i-1}} k + \sum_{k=1}^{2^{i-1}} 1 \right) \\ &= \pi t \sum_{i=1}^{n} \left(2^{2(i-1)} - \frac{2^{i-1}(2^{i-1}+1)}{2} + 2^{i-1} \right) \\ &= \pi t \sum_{i=1}^{n} \left(\frac{2^{2i-2} + 2^{i-1}}{2} \right) \\ &= \frac{\pi t}{8} \left(\sum_{i=1}^{n} 2^{2i} + 2 \sum_{i=1}^{n} 2^{i} \right) \end{aligned}$$

We know the following results:

$$\sum_{i=1}^{m} a^{i} = a\left(\frac{a^{m}-1}{a-1}\right) \quad \text{and}$$

$$\sum_{i=1}^{m} a^{2i} = \frac{a^{2}}{a+1}\left(\frac{a^{2m}-1}{a-1}\right)$$

Here we have $a = 2$. Hence:

$$\begin{aligned} L &= \frac{\pi t}{8}\left(\frac{4}{3}\left(2^{2n}-1\right) + 4\left(2^{n}-1\right)\right) \\ &= \frac{\pi t}{2}\left(\frac{2^{2n}-1}{3} + 2^{n} - 1\right) \\ &= \frac{\pi t}{6}\left(2^{2n} + 3\cdot 2^{n} - 4\right) \\ \therefore\ L &= \frac{\pi t}{6}\left(2^{n}+4\right)\left(2^{n}-1\right). \end{aligned}$$

Observe that L is expressed in terms of a quadratic equation in 2^n. Britney found it interesting to realise that when we fold a piece of paper, we are actually finding a solution to a quadratic equation!

Realising the theoretical limit

On 27th January 2002, after eight hours of hard work by three people, a tissue paper (4000 feet long, 0.0033 inches thick) was folded twelve times in half for the first time in Pomona, California. (See Figure 3.) This showed that the commonly held belief that it is not possible to fold a piece of paper in half more than eight times was false!

But note that from the above formula, we can say that approximately 2417 feet of tissue paper (of thickness 0.0033 inch) would have been enough to accomplish the task of folding paper in half twelve times. For a nice writeup on this episode, please see [5]. See also [6].

Figure 3. Britney Gallivan holding the first sheet of paper ever to be folded twelve times [©Britney C. Gallivan]; source: [1], [4]

Exercise

After the above analysis, the following question may arise: "How many times can we fold a sheet of paper in half, by folding in alternate directions, keeping all other rules the same?"

As expected, the answer is: *The length and width of the given sheet decide the number of times we can fold the paper in half*.

A good idea to proceed will be to start with a square sheet of paper and calculate its limiting width. Without considering the effects of material lost in radii of earlier folds, we get a crude bound for the width W of a square sheet of paper of thickness t required for folding in half n times as:

$$W = \pi t 2^{3(n-1)/2}.$$

I invite readers to derive the above formula.

Surely, the above formula does not give any minimum limit. Using analysis seen in *one direction folding*, we can derive a *Limit Formula* for *alternate folding*. But in *alternate folding*, we get separate equations for odd and even folds. Note that odd folds accumulate losses in an odd fold direction, and even folds accumulate losses in an even fold direction. Also, each fold in an odd direction doubles the amount of paper for the next even direction, and vice versa.

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in the classroom

Flashback to the past

A 1949 Matric Geometry Question

Michael De Villiers

Recently I was paging through my copy of the August 1996 issue of the unfortunately now defunct journal *Spectrum 34(3)*, and a section on old Mathematics Papers on p. 63 where the following problem from Paper 2 of the 1949 National Senior Examinations for the Union of South Africa caught my attention:

"In a quadrilateral *ABCD*, angles *B* and *C* are right angles. A straight line *EF* is drawn perpendicular to *AD*, and cuts *AD* and *BC* internally at *E* and *F* respectively. Prove that ∠*BEC* = ∠*AFD*."

Before continuing further, the reader is now urged to first try and prove the result.

For those of a younger generation, it may come as a surprise that like this question, none of the geometry questions included a sketch or diagram, as is customary in matric exams today. At the time it was expected of pupils (learners) to read, interpret and be able to draw their own diagrams from such a verbal, symbolic description. Even when I was in high school in the early seventies, this was still the case, but this was changed some time in the late seventies or early eighties to probably try and make things easier, not only for pupils (learners), but probably also for examiners in not having to decipher pupils' (learners') rough drawings when marking.

It should be noted, however, that in high-level mathematical competitions such as the Senior Third Round of the South African Mathematical Olympiad (SAMO) as well as the International Mathematical Olympiad (IMO), it is still customary that no diagrams are provided for learners, and that they are required to make their own drawings. So at this level, interpreting a verbal description of a geometric problem, and making an appropriate representation, is seen as part and parcel of mathematical competence and creativity. See for example, some past SAMO and IMO papers at the following two websites respectively:

http://www.samf.ac.za/QuestionPapers.aspx
http://www.imomath.com/index.php?options=924&lmm=0

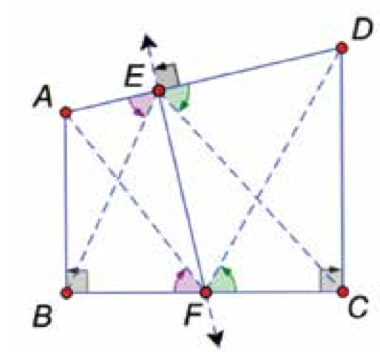

Figure 1

Proof

The proof is no easier or more difficult than ones currently expected from Grade 11/12 learners. Consider Figure 1. From the given construction, it follows that *ABFE* and *DCFE* are both cyclic quadrilaterals, since ∠*AEF* = ∠*DCB* and ∠*DEF* = ∠*ABC*, respectively. Therefore, on chord *AB*, ∠*AEB* = ∠*BFA* and on chord *DC*, ∠*DEC* = ∠*CFD*. It now follows that, ∠*BEC* = 180º – ∠*AEB* - ∠*DEC* = 180º – ∠*BFA* – ∠*CFD* = ∠*AFD*, which completes the proof .

Can we prove it differently? As is most often the case, another easy way to prove the result comes from proving the similarity of triangles *AFD* and *BEC* (since corresponding ∠*EAF* = ∠*EBF* on chord *EF* in cyclic *ABFE* and ∠*EDF* = ∠*ECF* on chord *EF* in cyclic *DCFE*). Note that we often prove mathematical theorems in different ways, not because we feel a need to verify the validity of the result, but *merely to increase our understanding or to look at it in a* different way (De Villiers, 1990).

Further Reflection

Pólya (1945) has most famously emphasized that the fourth step in problem solving is that of looking back, and reflecting on one's solution. Unfortunately this aspect of problem solving is seldom highlighted in teaching: usually the teacher and the children are in a hurry and just happy to move on to the next problem. However, reflecting on a solved problem or proof can often lead not only to deeper understanding and appreciation of a particular result, but also show how mathematics is sometimes developed further.

First, note that the result is really about a trapezium *ABCD* with *AB* and *DC* parallel (both perpendicular to *BC*). Secondly, looking back at the proof, it should be clear that the result depends on *ABFE* and *DCFE* both being cyclic quadrilaterals. But is it really necessary that the angles at *B* and *C* are right angles?

Clearly not, as all that is required is that the following remains true: ∠*AEF* = ∠*DCB* and ∠*DEF* = ∠*ABC*. Since ∠*AEF* + ∠*DEF* = 180º, it follows that the only requirement is that ∠*DCB* + ∠*ABC* = 180º; in other words, that *AB* has to be parallel to *DC*. In other words, the result is true for any trapezium *ABCD* with *AB* // *DC* and line EF constructed so that ∠*AEF* = ∠*DCB* (or equivalently ∠*DEF* = ∠*ABC*), as shown in Figure 2. It is left to the reader to verify that exactly the same proof applies. This is therefore an example of what has been called the *discovery* function of proof (De Villiers, 1990), whereby further analyzing the conditions of a proof can sometimes lead to further generalizations.

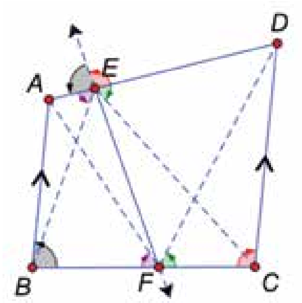

Figure 2

What if?

'Doing mathematics' is not just about answering questions, but also about asking questions, investigating and posing one's own problems. In this regard, an important mathematical habit of the mind is to ask 'what-if?' questions. Though the original problem was restricted to E and F on the 'interior' of segments AD and BC respectively, an obvious 'what-if' question to ask is: what happens when these points lie on the extensions of AB and BC? The reader is now invited to dynamically explore this situation with the following interactive sketch online:

http://dynamicmathematicslearning.com/matric-exam-1949.html

As the reader would've found, the equality of the two angles remains true even when E and F are moved outside onto the extensions of segments AB and BC. The result follows just as before, but with some variation. Consider, for example, the situation as shown in Figure 3 where both E and F lie on the extensions of the two segments.

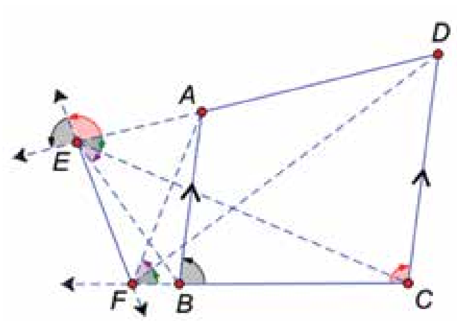

Figure 3

As before, it follows that $ABFE$ and $DCFE$ are both cyclic quadrilaterals, since $\angle AEF = \angle ABC$ and $\angle DEF + \angle DCF = 180^o$, respectively. Therefore, on chord AB, $\angle AEB = \angle BFA$ and on chord DC, $\angle DEC = \angle CFD$. It now follows that $\angle BEC = \angle AEB - \angle DEC = \angle BFA - \angle CFD = \angle AFD$, which completes the proof. It's now left to the reader to verify that the result also holds for the other cases if E and F lie on the extensions of AD and BC towards the opposite sides, or when one of E or F lies on a segment while the other lies on an extension.

Since there are several cases to consider, it is probably for this reason that the examiners decided to simplify the problem by restricting it to the interior of the trapezium, but in the process the interesting generality of the result was unfortunately lost. It should be noted, however, that if one uses the advanced concept of 'directed angles', or better still, use vector or complex algebra, it is possible to give a general proof that covers all cases (including the crossed case discussed below).

Though the examiners probably only had a convex trapezium in mind (assuming perhaps implicitly that BA and CD were on the same side of BC), another configuration not explicitly excluded by the examiners is the case when the trapezium becomes crossed as shown in Figure 4. Does the result still hold for the crossed trapezium $ABCD$ shown?

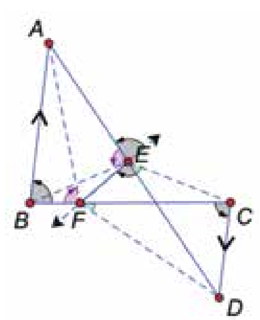

Figure 4

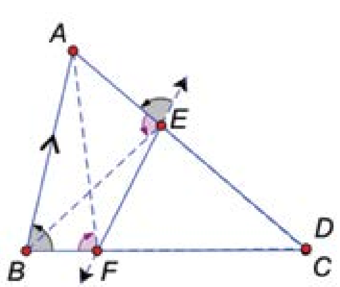

Figure 5

As can easily be verified experimentally by the reader, the result is still valid. For example, simply drag the point D in the dynamic geometry sketch in the link given earlier, in the opposite direction until it moves past C as shown in Figure 4,

However, the previous proofs require a little modification for the crossed case. As before, $ABFE$ is still cyclic since the exterior $\angle FED = \angle ABF$ (by construction). Since AB // CD, alternate angles ABF and FCD are equal; and $\angle FED = \angle FCD$; hence $EFDC$ is also a cyclic quadrilateral (equal angles on chord FD). Therefore, as before, on chord AB, $\angle AEB = \angle BFA$, and on chord DC, $\angle DEC = \angle CFD$. It now follows that:

$\angle BEC = 180^o - \angle AEB + \angle DEC = 180^o - \angle BFA + \angle CFD = \angle AFD$

Also note, as shown in Figure 5, that angles BEC and AFD remain equal in the special case when, say, C and D coincide; in other words when the trapezium degenerates into a triangle.

Alternative formulations

Since the problem involves cyclic quadrilaterals, we could instead of starting with a trapezium, consider what happens if we start with two intersecting circles P and Q as shown in Figure 6. If we construct straight lines AED and BFC through the two intersections of the circles as shown, we obtain AB // CD, and as before $\angle BEC = \angle AFD$. In a sense, this variation can be considered as a type of converse of the original result since the premise here is the intersecting circles (i.e. implicitly the cyclic quadrilaterals $ABFE$ and $EFDC$) with the conclusion now that $ABCD$ is a trapezium with AB // CD (whereas in the original general version formulated with respect to Figure 2, this conclusion is assumed as given). It is left to the reader to verify that the result holds in the case illustrated (as well as other possible configurations such as a crossed configuration). Another interesting converse-like formulation of the result is the following. Given the two circles,

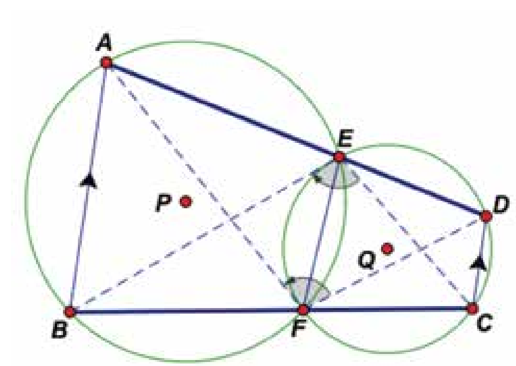

Figure 6

if AB is drawn first, followed by the straight line BFC, and CD is then drawn parallel to BA, then D, E and A are collinear (lie in a straight line). The proof of this is similar to those before, and is also left as an exercise to the reader.

Maximizing the angle

If the reader goes back to the original dynamic sketch, and drags the moveable point E, it will be immediately noticeable that $\angle BEC$ is a variable. This now raises another interesting question, namely, whether the angle has a maximum and where to locate it.

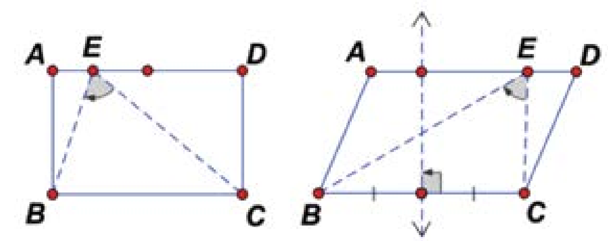

Figure 7

As is well known in problem solving, it often helps to understand a problem better by looking at special cases. If we consider a rectangle as a special case of a trapezium as shown in the first figure in Figure 7, it is obvious from symmetry that the maximum value of angle BEC will be at the midpoint of AD. In the case of the trapezium being a parallelogram, it's not hard to intuitively 'see' (or find experimentally with a dynamic geometry sketch) that the optimal value of angle BEC is obtained when E is located at the intersection of the perpendicular bisector of BC and the opposite side DA.

What about the general case when $ABCD$ is a trapezium with AB // CD? Remembering the analogous problem of maximizing the kicking angle in rugby as discussed in De Villiers (1999), it was clear to me that the solution would similarly lie where the circle through B and C touched the line AD.

Consider Figure 8. It is now easy to prove as it follows that angle BEC would be maximized when E is placed at the tangent point, G, where the circle through B and C touches the line AD. Since they are on the same chord BC, any inscribed $\angle BHC = \angle BGC$. But from the exterior angle theorem, it follows that $\angle BHC > \angle BEC$; hence $\angle BEC$ is a maximum only when E is placed at G.

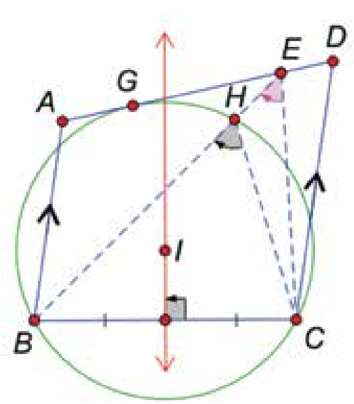

Figure 8

One can easily find this tangent point *G* experimentally using dynamic geometry by first constructing the perpendicular bisector of *BC*. Then locating a centre *I* of the circle on the perpendicular bisector, one can move the point *I* up and down until the circle touches the line *AD*. However, this is not very satisfactory as apart from being prone to 'experimental error', it is dependent entirely on a particular trapezium *ABCD*, and as soon as the trapezium's shape is changed, one will have to repeat the same experimental approach of moving the circle. So ideally, we'd like to have a way of precisely locating the centre *I* of the tangent circle that gives the optimal position of *E*. How can that be done?

Think about it! We are looking for a point that is equidistant from the points *B* and *C* as well as the line *AD*. Since a perpendicular bisector is the locus of all equidistant points from *B* and *C*, the point *I* clearly must lie somewhere on the perpendicular bisector of *BC*, but exactly where?

Recall a result already known to the ancient Greeks that the locus of all points equidistant from a point (called a focus) to a line (called the directrix) is a parabola[1]! Therefore, all we now need to do is use a standard dynamic geometry facility to construct the parabola determined by point *B* as focus and line *AD* as directrix, and where this parabola meets the perpendicular bisector, we have the required point *I* that is equidistant from line *AD* as well as points *B* and *C* as shown in Figure 9. *Voila!*

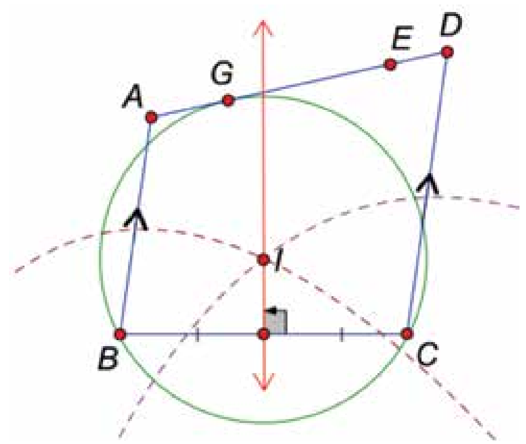

Figure 9

Also note that from the construction above, since *I* is equidistant from points *B* and *C* as well as line *AD*, the point *I* also lies on the parabola determined by *C* as focus and *AD* as directrix. Hence this also proves the interesting, unusual result of two parabolas and a line concurrent in a point, something which learners at high school are usually not exposed to. Also note that since the result only involves the line *AB* and the points *B* and *C*, it is not specific to a trapezium any more, but applies to determining the maximum of $\angle BEC$ in any quadrilateral *ABCD* with *E* on line *AD*. With the increasing availability of dynamic geometry software in classrooms around the world, this general problem and result can easily be tackled and solved at high school.

In addition, the optimization problem about the angle is more elegantly solved without the use of calculus, purely with synthetic geometry. It can therefore show high school learners the value of geometry, and that one need not always resort to algebra and calculus to minimize or maximize a particular variable.

[1]*For more information on parabola, go to: https://en.wikipedia.org/wiki/Parabola Or see my paper "Exploring loci on Sketchpad" at: http://www.researchgate.net/publication/242188586_Exploring_Loci_on_Sketchpad*

Concluding comments

Starting with a standard matric (Grade 12) problem from years ago, which on the surface may even appear rather routine and boring, this paper has shown how further reflection and exploration of the original result has actually revealed a very rich problem leading to generalizations, alternative formulations and an optimization problem. Sometimes exploring one problem in more depth like this one can indeed be a better educational experience for learners than doing numerous examples, but only covering them superficially.

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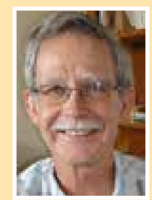

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in the classroom

Triangle Centres in an Isosceles Triangle

A Ramachandran

Every triangle has certain lines associated with it. The most prominent among them are the perpendicular bisectors of the sides, the bisectors of the angles, the altitudes, and the medians. Figure 1 represents a scalene triangle ABC, with $AB < AC$. Also shown are the altitude AD from A to BC, the bisector AE of angle A, the median AF where F is the midpoint of BC, and the perpendicular bisector of BC.

We must justify the order in which these lines appear in the figure: the altitude is the closest to AB (the shorter of the sides AB and AC), then the angle bisector, followed by the median, and the perpendicular bisector is closest to side AC. It is of interest to see whether this ordering can be justified using the regular results of Euclidean geometry. Indeed it can, and here's how:

- $\triangle ADB$ and $\triangle ADC$ are right angled. Also, $\measuredangle ABC > \measuredangle ACB$, hence $\measuredangle BAD < \measuredangle CAD$ and therefore $\measuredangle BAD < \frac{1}{2}\measuredangle BAC$,

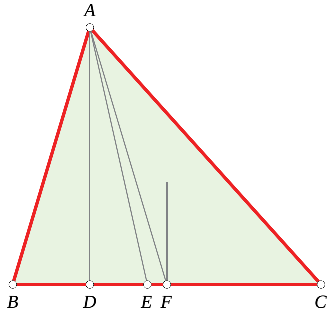


Figure 1. Four significant lines: altitude, angle bisector, median, perpendicular bisector

Keywords: Triangle, circumcentre, incentre, orthocentre, centroid, Euler line

i.e., $\measuredangle BAD < \measuredangle BAE$. Therefore the altitude lies between AB and AE. Hence D lies between B and E.

- The angle bisector theorem tells us that angle bisector AE divides the base BC in the ratio $AB : AC = c : b$. Since $AB < AC$, it follows that $BE < EC$ and therefore that $BE < \frac{1}{2}BC$. Hence E lies between B and F.
- In $\triangle ABF$ and $\triangle ACF$ we have: $BF = FC$, and AF is a shared ('common') side. Since $AB < AC$, it follows that $\measuredangle AFB < \measuredangle AFC$. Therefore the perpendicular to BC at F lies to the right of median AF.

Note: We are using here the "inequality form of the SAS congruence theorem." We state it with reference to two triangles PQR and LMN (see Figure 2).

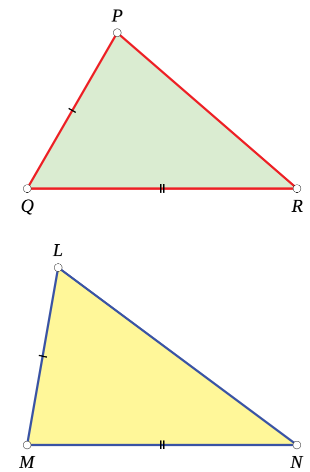


Figure 2. Inequality form of the SAS congruence theorem

Suppose that $PQ = LM$ and $QR = MN$. Then we have the following:

⋆ if $\measuredangle Q < \measuredangle M$, then $PR < LN$;

⋆ if $PR < LN$, then $\measuredangle Q < \measuredangle M$.

Note that the second part is the converse of the first part.

If $AB = AC$, the four lines discussed above (altitude, angle bisector, median and perpendicular bisector of side) merge into a single line of symmetry of the triangle, with

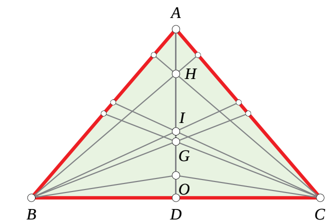


Figure 3. The case of an isosceles triangle

D, E, F coinciding (see Figure 3). That is, these three points are either all distinct or all coincident. The corresponding lines associated with the other two sides of the triangle continue to be distinct unless $AB = AC = BC$.

It is well known that the altitudes of a triangle are concurrent at the ***orthocentre*** (generally denoted by the letter H), the angle bisectors at the ***incentre*** (I), the medians at the ***centroid*** (G) and the perpendicular bisectors of the sides at the ***circumcentre*** (O). These four "triangle centres" are distinct points in a scalene triangle. (It will be a nice exercise for you to prove that if any two of the points I, O, G, H coincide, then they all coincide.)

In any triangle the points H, G, O are collinear, as shown by Leonhard Euler in 1765. The line of collinearity is called the ***Euler line*** of the triangle, and G lies between H and O on this line, dividing HO in the ratio $2 : 1$. (The point I in general does not lie on the Euler line, unless the triangle is isosceles.) In an equilateral triangle the three points merge into a single point. In an isosceles triangle such as $\triangle ABC$, with $AB = AC \neq BC$ (Figure 3), they are distinct and lie on the line of symmetry AD, which is also the Euler line for the triangle.

From this point on we shall confine our discussion to the case of an isosceles triangle ABC in which $AB = AC$. Let D be the midpoint of BC; then AD is a line of symmetry for the triangle. The claim that the points O, G, I, H all lie on AD is easy to justify, using the well-known theorems of congruence. We claim that the points always occur in the order O, G, I, H on the line. For the proof, use will be made of the fact that in any triangle, the centroid G lies 2/3 the way along each of the medians. So the ratio $AG : AD$ equals $2 : 3$, regardless of the shape of the triangle. (See page 52 of the

November 2013 issue of *At Right Angles* for a proof of this assertion.)

More specifically, we claim the following:

- If $\measuredangle A < 60°$, then $BC < AB = AC$, so H lies closest to BC, followed by I, G and O, in that order.
- If $\measuredangle A > 60°$, the order gets reversed, since now $BC > AB = AC$. (Of course, when $\measuredangle A = 60°$, the four points are coincident.)
- If $\measuredangle A = 90°$, then H coincides with A, while O coincides with the midpoint D of BC. Observe that in this configuration the fact (Euler's theorem) that the ratio $HG : GO$ equals $2 : 1$ reduces to the known fact that the centroid lies 2/3 the way along a median.
- If $\measuredangle A > 90°$, then both H and O lie outside the triangle.

To justify the first two claims, we derive expressions for the distances AH, AI and AO, as fractions of the altitude AD, as $\measuredangle A$ varies.

With reference to Figures 4 and 5, we have:

$$\frac{HD}{BD} = \tan\frac{A}{2}, \quad \frac{BD}{AD} = \tan\frac{A}{2}, \tag{1}$$

hence:

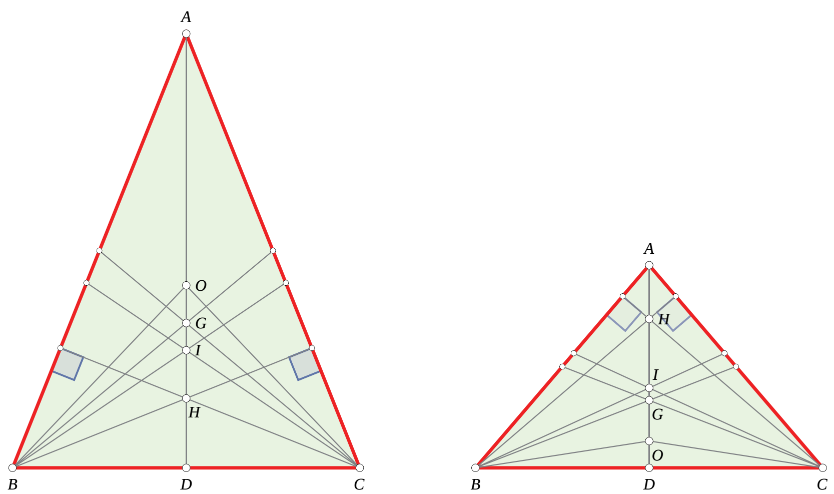
$$\frac{HD}{AD} = \tan^2\frac{A}{2}. \tag{2}$$

Next:

$$\frac{ID}{BD} = \tan\frac{B}{2} = \tan\left(45° - \frac{A}{4}\right), \tag{3}$$

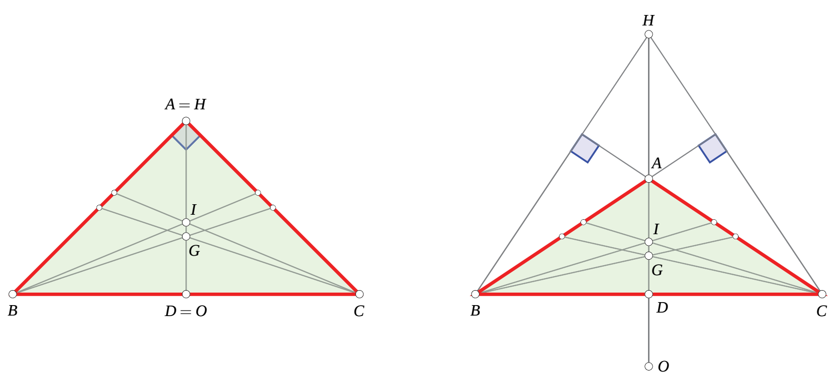
so:

$$\frac{ID}{AD} = \frac{ID}{BD}\cdot\frac{BD}{AD} = \tan\frac{A}{2}\cdot\tan\left(45° - \frac{A}{4}\right). \tag{4}$$



When $\angle A < 60°$ When $\angle A > 60°$

Figure 4. Isosceles triangles with different apex angles



When $\angle A = 90°$ When $\angle A > 90°$

Figure 5. Isosceles triangles with different apex angles

The ratio for G is easy:

$$\frac{GD}{AD} = \frac{1}{3}. \tag{5}$$

Finally, for O we have: $\measuredangle BOC = 2\measuredangle A$, therefore $\measuredangle OBD = 90° - A$. This yields:

$$\frac{OD}{BD} = \tan(90° - A) = \frac{1}{\tan A},$$

hence:

$$\frac{OD}{AD} = \frac{\tan\frac{1}{2}A}{\tan A}. \tag{6}$$

Using the double-angle formula to express $\tan A$ in terms of $\tan\frac{1}{2}A$, the above expression for the ratio $OD : AD$ may be written more usefully as:

$$\frac{OD}{AD} = \frac{1 - \tan^2\frac{1}{2}A}{2}. \tag{7}$$

From the above relations we see that

$$\frac{HD}{AD} + 2\cdot\frac{OD}{AD} = 1,$$

and hence:

$$\boxed{\frac{1}{3}\cdot\frac{HD}{AD} + \frac{2}{3}\cdot\frac{OD}{AD} = \frac{GD}{AD}}. \tag{8}$$

This directly shows that G lies between O and H and divides segment OH in the ratio $1 : 2$.

It is an easy exercise to verify that if $\measuredangle A = 60°$ then

$$\frac{HD}{AD} = \frac{ID}{AD} = \frac{GD}{AD} = \frac{OD}{AD} = \frac{1}{3}.$$

The relative order of the four points H, I, G, O on AD. Observe that if $A < 60°$ then $\frac{3}{4}A < 45°$ and so $\frac{1}{2}A < 45° - \frac{1}{4}A$. It follows that if $A < 60°$ then $\tan\frac{1}{2}A < \tan\left(45° - \frac{1}{4}A\right)$ and hence from relations (2) and (4) that

$$\boxed{\frac{HD}{AD} < \frac{ID}{AD}}.$$

The inequality is reversed when $A > 60°$.

Now let us compare the relative positions of I and G. This involves more manipulations than the other cases. We have:

$$\begin{aligned}\frac{ID}{AD} &= \tan\frac{A}{2}\cdot\tan\left(45° - \frac{A}{4}\right)\\ &= \frac{2\tan\frac{1}{4}A}{1 - \tan^2\frac{1}{4}A}\cdot\frac{1 - \tan\frac{1}{4}A}{1 + \tan\frac{1}{4}A} = \frac{2\tan\frac{1}{4}A}{\left(1 + \tan\frac{1}{4}A\right)^2}\\ &= \frac{2t}{(1+t)^2}, \quad \text{where } t = \tan\frac{A}{4}.\end{aligned}$$

Since $0° < A < 180°$, it must be that $0 < \tan\frac{1}{4}A < 1$, i.e., $0 < t < 1$. Differentiation yields:

$$\frac{d}{dt}\left(\frac{2t}{(1+t)^2}\right) = \frac{2(1-t)}{(1+t)^3},$$

which is positive for $0 < t < 1$. Hence the expression $2t/(1+t)^2$ steadily increases as t goes from 0 to 1. Therefore the quantity

$$\frac{ID}{AD} = \frac{2\tan\frac{1}{4}A}{\left(1 + \tan\frac{1}{4}A\right)^2}$$

steadily increases as A rises from 0° to 180°. Simple computation shows that the above fraction equals 1/3 when $A = 60°$. (We need the identity $\tan 15° = 2 - \sqrt{3}$.) It follows that

$$\frac{ID}{AD} < \frac{1}{3} \quad \text{if } \measuredangle A < 60°,$$

$$\frac{ID}{AD} > \frac{1}{3} \quad \text{if } \measuredangle A > 60°.$$

We conclude from this that I always lies between H and G for an isosceles triangle, and the four points H, I, G, O occur in that order always, with O being closer to vertex A when $\measuredangle A < 60°$, and H being closer to vertex A when $\measuredangle A > 60°$.

The various possibilities which may exist for the order of the points O, G, H, I on the Euler line in the case of an isosceles triangle ABC with $AB = AC$ are summarised in Figure 6.

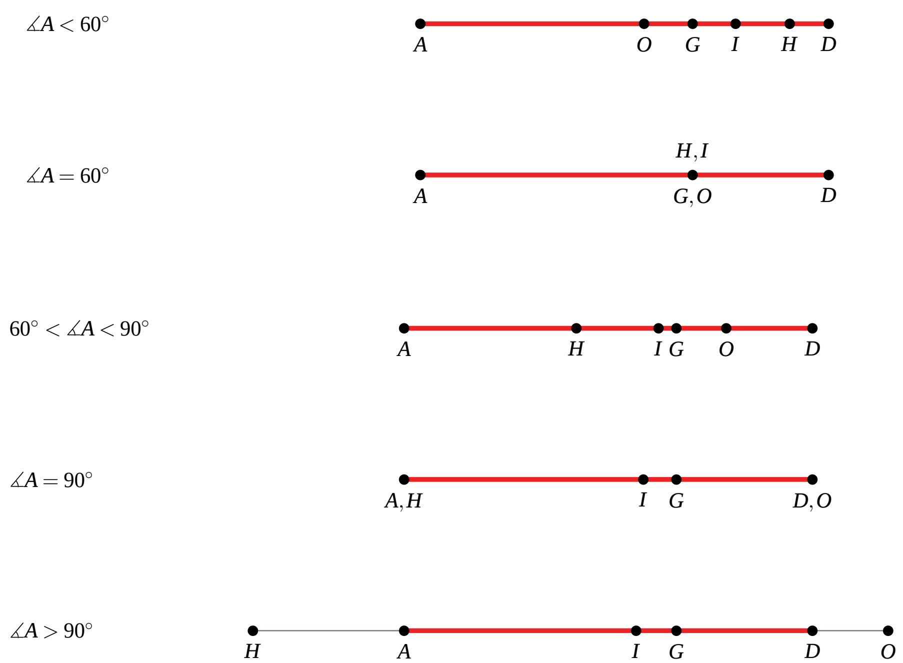
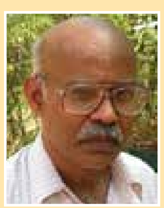


Figure 6.



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Low Floor High Ceiling Tasks

Getting into Shape

Tangram Time

Sneha Titus & Swati Sircar

In the November 2014 issue of *At Right Angles,* we began a new series which was a compilation of 'Low Floor High Ceiling' activities. A brief recap: such an activity comprises a sequence of tasks which are fairly easy to begin with and can be attempted by all the students in the class. However, the tasks progressively become more difficult. The objective is to challenge the problem-solving skills of students and in attempting them, each student is pushed to his or her maximum potential. There is enough work for all but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task. In the first part of the series (in the November 2014 issue), we looked at pentominoes and in the March 2015 issue at the Fibonacci series and the regular pentagon. This time we turn to an old favourite: tangrams!

Keywords: *tangram, triangle, quadrilateral, square, parallelogram, rectangle, congruence, similarity, collaboration*

This is how Wikipedia describes this popular puzzle. The **tangram** (Chinese: 七巧板; pinyin: *qīqiǎobǎn*; literally: "seven boards of skill") is a dissection puzzle consisting of seven flat shapes, called *tans*, which are put together to form shapes. The objective of the puzzle is to form a specific shape (given only an outline or silhouette) using all seven pieces, which may not overlap. It is reputed to have been invented in China during the Song Dynasty, and then carried over to Europe by trading ships in the early 19th century. It became very popular in Europe for a time then, and then again during World War I. It is one of the most popular dissection puzzles in the world. A Chinese psychologist has termed the tangram "the earliest psychological test in the world", albeit one made for entertainment rather than analysis [1].

Tangram tasks are appropriate for students of class 6 and upwards. As usual, each card (or set of cards) is a task which features a series of questions which build up in complexity. Since concepts such as congruence or similarity are dealt with only from class 7 onwards, it is possible that the teacher may have to select tasks which are appropriate for the students. However, these activities also provide a platform for a gentle and informal introduction to concepts taught at a higher grade.

TASK 1: To identify and describe the 7 Tangram pieces separately

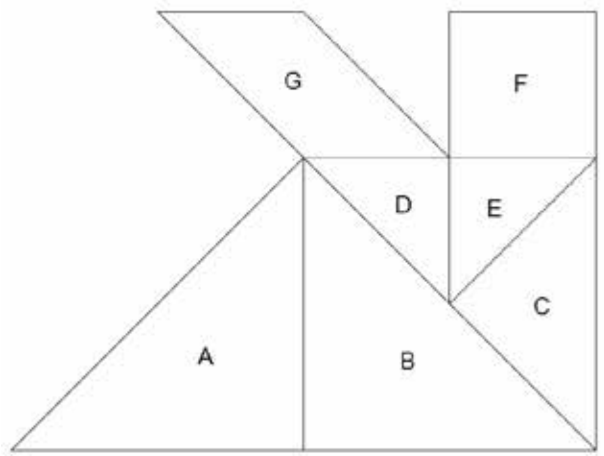


Fig. 1

Look at this tangram set (Fig. 1).

- Place tracing paper over this set and make cutouts of all the 7 shapes (A-G).
- Arrange the cutouts in sets of similar shapes.
- Label each set and describe the properties common to the members of each set.
- If there is more than one piece in a set, classify this set into subsets of congruent shapes. Describe the relationship between the subsets of the shapes.
- Write down your observations.

Teacher's Note: This is an easy introduction to the tangram kit for students who are not familiar with it. The aim of this task is primarily mathematical communication and documentation. It also helps students review and distinguish between terms such as 'congruent', 'similar', 'set', 'subset', 'members', etc. Knowledge and understanding of the basic angles and shapes is a necessary prerequisite for this task. The terms 'congruence' and 'similarity' may be replaced by 'replica' and 'scaled up/down version' respectively.

TASK 2: To reassemble the Tangram kit and describe the kit as a whole

For this task, the class may be divided into two groups. Each group is given one of the pictures shown (either Fig. 2.1 or Fig. 2.2) and not allowed to look at the other picture. Instructions for both groups remain the same except where specified.

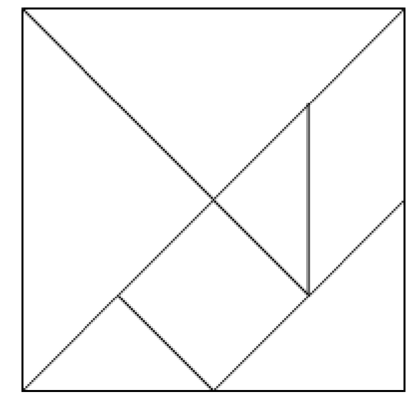


Fig. 2.1

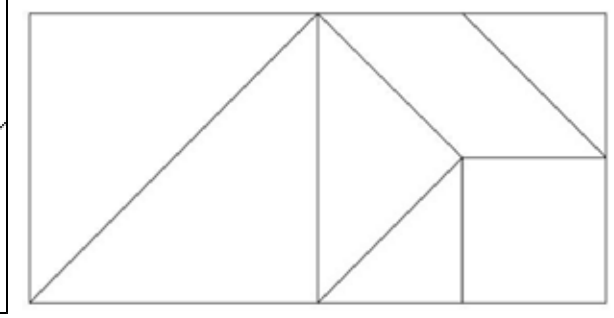


Fig. 2.2

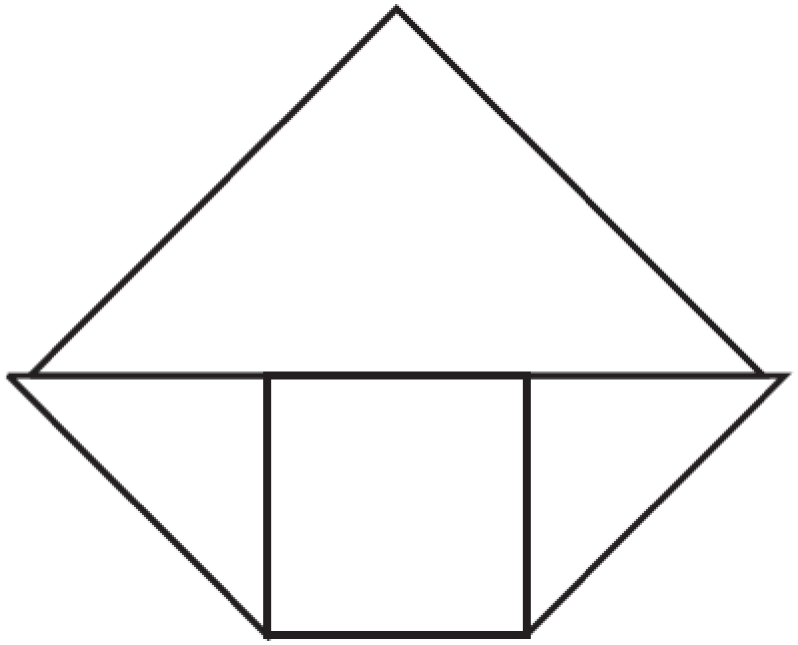
- Re-assemble your cutouts to make the picture your group was given. Identify the final assembled quadrilateral. Label the polygons A-G with the same labels given in Task 1.
- Copy your quadrilateral (Fig. 2.1 or Fig. 2.2), with all the parts A-G fitting, into your notebook. Write down a set of instructions with which this picture can be re-created by someone who hasn't seen it before.
- Group 1: Working with a partner from group 2, dictate your set of instructions to your partner and check if he/she was able to draw Fig. 2.1.
- Group 2: Now, working with the same partner from group 1, dictate your set of instructions to your partner and check if he/she was able to draw Fig. 2.2.
- You have identified the geometrical shapes A-G in Task 1 and described their properties. Discuss with your partner and write down why the instructions given produced the shapes A-G and made them fit together.

Teacher's Note: The objective of this task is for students to see how the 7 pieces of the tangram fit together to make a square or a rectangle. Once either quadrilateral is assembled, they also have the opportunity to see how it can be deconstructed into the original 7 pieces. The instructions they write should produce the very same shapes A-G. Labelling the four vertices of the square or rectangle as well as new points on the sides (beginning from H to avoid confusion) will bring better clarity to the instructions and may be suggested to the students. Here is the opportunity for peer assessment of mathematical communication skills. Feedback is instant and corrective measures can be suggested by peers to improve work.

For those with access to dynamic geometry software, such as GeoGebra, this can also work as a lab exercise to create the sketch with the instructions that have been noted down. The last question is definitely a stab at the high ceiling and may not be attempted by all, as it requires students to use properties such as: the diagonals of a square intersect at right angles. But it sets the foundation for developing the skills of proof. After the attributes of each of the 7 shapes are checked using either the compass box or GeoGebra, students can practice their reasoning, logic and knowledge of geometry by explaining why the instructions they gave produced these shapes.

TASK 4: To study the lengths of the sides of different shapes and to use Pythagoras' theorem.

- Taking the side of the square as unit length, find the lengths of the sides of the small triangle and the biggest triangle.
- Express the hypotenuse of the big triangle in 2 different ways.
- Examine and re-create the following picture with your tangram pieces:



Why is the hypotenuse of the big triangle slightly less than 3 times the sides of the square?

Teacher's Note: A necessary prerequisite for this task is the knowledge of Pythagoras' theorem and the understanding and manipulation of irrational numbers. Students should be guided to arrive at $\sqrt{8} = 2\sqrt{2}$ which is the purpose of stressing on two different ways to arrive at the hypotenuse. In the second part, students get practice in approximating the value of an irrational number, which is also an exercise in mathematical communication, reasoning and logic.

TASK 4.1: To assemble the individual pieces into geometrical shapes.

- In how many ways can you make a square i.e. with
  i. 1 piece
  ii. 2 pieces ....and so on.

Draw each configuration.

Teacher Note: To make the square, there is only 1 possible configuration in each case with 1, 3, 4, 5 and 7 pieces. With 2 pieces, 2 configurations are possible where the square can be big or small with the respective pairs of identical triangles. This task may be done as a paired activity as students will benefit from discussion and will be able to share their understanding about the properties of each shape.

TASK 4.2

- Which of the following shapes: triangle, rectangle, parallelogram, isosceles trapezium and trapezium can you make using:
  i. Any of the pieces
  ii. Any 2 pieces
  iii. Any 3 pieces
  iv. Any 4 pieces
  v. Any 5 pieces
  vi. Any 6 pieces
  vii. All 7 pieces
- Draw each configuration and complete the following table

No. of pieces →	1	2	3	4	5	6	7
Triangle	✓						
Square	✓						
Rectangle	×						
Parallelogram							
Trapezium							
Isosceles trapezium							

Teacher's Note: For shapes other than the square, more than one configuration is possible with the same number of pieces. This task can also be assigned as group work as time constraints may restrict a student from coming up with all the options. Students are called upon to use the properties of each shape, most often intuitively rather than overtly. The table helps them put down their findings systematically.

TASK 5: To prove that a certain configuration is impossible

In Task 4, which shapes could you not make with 6 pieces?

Prove that it is impossible to make these shapes with any combination of 6 pieces.

Teacher's Note: Making a conjecture and then proving it is a sophisticated mathematical skill.

Conjecture: It is not possible to make a square or a right isosceles triangle with any 6 pieces of the tangram set.

Proof: Let us assume that the side of the square is of unit length. The areas of the 7 pieces are given in the following table. The total area = 1 + 2 × ½ + 1 + 2 × 2 + 1 = 8 square units. In the following table we enumerate different cases where a shape is made by leaving out one of the 7 pieces.

Piece	Sides	Area
Square	1	1
Smallest triangles	1, 1, √2	½
Medium triangle	√2, √2, 2	1
Biggest triangles	2, 2, 2√2	2
Parallelogram	1, √2	1

Piece left out	Area of any shape made with the remaining 6 pieces
Biggest triangle	8 − 2 = 6
Medium triangle, square, parallelogram	8 − 1 = 7
Smallest triangle	8 − ½ = 7½

Thus, if any combination of 6 pieces are used to create a square, it will have an area of 6, 7 or 7½, hence it will have sides of √6, √7 or √(7½) = ½√30.

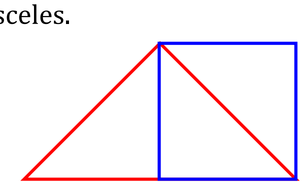
But with the pieces of the tangram set, we can only have sides of the form $a + b\sqrt{2}$ where a, b = 0, 1, 2, 3 ... (1)

This automatically rules out squares of area 6, 7 or 7½. Since neither √6 nor √7 or √30 can be expressed in the above form (1), we conclude that it is not possible to make a square with any 6 pieces.

The case of the triangle is very similar once we prove that it has to be right isosceles.

Any triangle must have 2 acute angles.

The only possible acute angle in any of the tangram piece is 45°.



∴ any triangle made of tangram pieces must have 2 angles of 45° each and therefore the 3rd angle must be 180° – (45° + 45°) = 90°

∴ it must be a right isosceles triangle.

Its area will be ½ base × height = ½ x^2 = 6, 7 or 7½ which means that x must be √12, √14 or √15 which as we saw is not possible with the pieces of the tangram set.

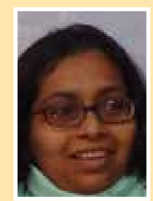
So we can conclude that it is not possible to make a triangle with any 6 pieces.

Conclusion

Tangrams have always been presented as fun and hands-on activities for the class and that is what we have aimed for with this set of low floor high ceiling tasks. But we have raised the bar with an added layer of proof which is aimed at developing students' reasoning, logical and communication skills. As always, the teacher should facilitate discussions but give the students time to develop their own line of thought - they are sure to surprise and delight you!

References

1. https://en.wikipedia.org/wiki/Tangram (downloaded on September 13, 2015)



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An impossible relation

In the accompanying article on *Tangrams*, a claim was made that it is not possible to find integers a and b which make any of the following equalities true:

$$\sqrt{6} = a + b\sqrt{2}, \qquad \sqrt{7} = a + b\sqrt{2}, \qquad \sqrt{12} = a + b\sqrt{2},$$

and so on. However, the proofs may not be obvious. In this brief note, we shall prove the impossibility of the first relation and leave the remaining ones for you to handle.

Claim. *It is not possible to find rational numbers a and b such that $\sqrt{6} = a + b\sqrt{2}$.* (Note that we have replaced the word 'integers' by 'rational numbers.' Thus we are proving a stronger version of the statement than the original one.)

Proof. As with the proofs of most assertions of this kind, this is a proof by contradiction. We shall assume that there do exist rational numbers a and b for which $\sqrt{6} = a + b\sqrt{2}$ and then show that this assumption leads to a contradiction.

As readers must be familiar with the well-known proof of the irrationality of $\sqrt{2}$, we shall not bother with repeating the proof. By using virtually the same reasoning, we can also prove the irrationality of the following numbers: $\sqrt{3}$ and $\sqrt{6}$. We shall assume that you have already gone through this exercise.

In the relation $\sqrt{6} = a + b\sqrt{2}$, it cannot be that $b = 0$, for this would mean that $\sqrt{6}$ is a rational number. Hence $b \neq 0$. It also cannot be that $a = 0$. For, if $a = 0$, then by division we would get $\sqrt{3} = b$, which would mean that $\sqrt{3}$ is a rational number. However, we know that this is not true. Hence $a \neq 0$. So both a and b are non-zero.

Squaring both sides of the relation $\sqrt{6} = a + b\sqrt{2}$, we get: $6 = a^2 + 2b^2 + 2ab\sqrt{2}$, hence:

$$\sqrt{2} = \frac{6 - a^2 - 2b^2}{2ab}.$$

In this relation, the denominator is non-zero, implying that $\sqrt{2}$ is a rational number. However, we know that this is not true. Hence the stated relation cannot hold. That is, it is not possible to find rational numbers a and b such that $\sqrt{6} = a + b\sqrt{2}$. □

— $\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$

in the classroom

Theorems on Magic Squares

Shailesh Shirali

As has been pointed out in the companion piece on the same topic elsewhere in this issue, magic squares have been a source of recreation and leisure from ancient times. There is something about the symmetry and patterns contained in such squares that carry great appeal. In this piece, we shall prove two simple results about 3 × 3 and 4 × 4 magic squares. (Such squares would also be called *third-order* and *fourth-order* magic squares respectively.) Considering the ease and elegance with which they can be proved, the results will only add further appeal to an already wonderful subject.

Terminology. A magic square of order n is an $n \times n$ array of distinct positive integers with the property that its rows, its columns and its two main diagonals all have the same sum. This sum is called the ***magic sum*** of the square, or the *magic constant* of the square. A ***line*** of a magic square is any row, any column or either of the two main diagonals of the square. A magic square of order n thus has $2n + 2$ such lines.

Structure of a Third-Order Magic Square

We shall prove the following simple and pleasing properties which are exhibited by any third-order magic square. Let s be the magic sum of such a square, and let m be the number in the central cell of the square (i.e., the number in row # 2 and column # 2).

Keywords: *Magic square, order, magic sum, line, arithmetic progression, symmetry*

Then we have:

- $s = 3m$;
- There are four distinct three-term arithmetic progressions (APs) within the square: (i) the numbers in the central row, (ii) the numbers in the central column, (iii) the numbers in each of the two diagonals.

Note that if we prove the first assertion, the second one gets proved as well. For, if the three numbers in any of the triples referred to above are a, m, b, then we have $a + m + b = s = 3m$ (by the first assertion), hence $a + b = 2m$, i.e., $m - a = b - m$. This proves that a, m, b form an AP.

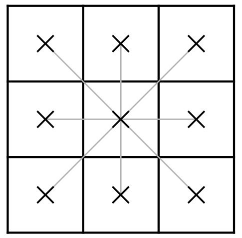

Figure 1

To prove the first assertion ($s = 3m$), we consider the four lines going through the central square (see Figure 1). The sum of the three numbers on each line is s, hence the four lines together yield a sum of $4s$, with some numbers getting counted more than once. Now note that the lines pass through every cell in the array, but the central square (which lies on all the lines) gets 'covered' four times. So the lines cover every cell in the whole array, with the central square getting covered an 'extra' three times. The sum of all the numbers in the whole array is equal to $3s$. These facts together lead to the following equation:

$$4s = 3s + 3m.$$

Hence $s = 3m$, as claimed.

Here are some notable consequences of this result: if the numbers in a third-order magic square are the numbers from 1 to 9, then the number in the central cell is necessarily 5. For, the sum of the numbers from 1 to 9 is 45, so the magic sum is $s = 45/3 = 15$. This yields $m = 5$.

Next, using the numbers from 1 to 9, the only three-term arithmetic progressions with central term 5 and sum 15 are {1, 5, 9}, {2, 5, 8}, {3, 5, 7} and {4, 5, 6}. These four APs correspond (in some order) to the four lines that pass through the central cell of the square. Now consider the lines that contain 1. Since the total of the three numbers in any line is 15, the sum of the other two numbers in such a line must be 14. The pairs which yield a sum of 14 are the following: {5, 9}, {6, 8}. Observe that there are only two such pairs. This implies that 1 cannot occur in a corner of the array (for that would require three pairs). Hence 1 occurs in the middle of a border row or column. By applying a suitable rotation, we can bring the 1 to the top row; this will naturally not disturb the 'magic' property. The two lines of which 1 is a part must have the pairs {5, 9} and {6, 8} for the remaining two numbers. Of these, the former pair corresponds to the line going through the centre of the square. The 6 and 8 may be filled in the top corner cells in either order. Once this is done, the remaining cells get filled on their own. The result is shown in Figure 2.

8	1	6
3	5	7
4	9	2

Figure 2

Structure of a Fourth-Order Magic Square

Fourth-order magic squares have rather more complex symmetries than third-order magic squares. One such symmetry is indicated in Figure 3. We shall prove that if p, q, r, s are the numbers in the cells as indicated, then the following equality necessarily holds:

$$p + q = r + s.$$

p			q
	r	s	

Figure 3

To prove the claim, it is convenient to use the symbols shown in the array below.

a_1	b_1	c_1	d_1
a_2	b_2	c_2	d_2
a_3	b_3	c_3	d_3
a_4	b_4	c_4	d_4

We must prove that $a_1 + d_1 = b_4 + c_4$. We use repeatedly the defining property of a magic square. The sum of the numbers in the two main diagonals equals the sum of the numbers in the two middle columns, hence:

$$(a_1 + b_2 + c_3 + d_4) + (a_4 + b_3 + c_2 + d_1) = (b_1 + b_2 + b_3 + b_4)+(c_1 + c_2 + c_3 + c_4) .$$

On cancellation of common terms, this simplifies to:

$$a_1 + d_4 + a_4 + d_1 = b_1 + b_4 + c_1 + c_4.$$

We rewrite this as follows:

(1) $$a_1 + d_1 - b_1 - c_1 = b_4 + c_4 - a_4 - d_4.$$

We also have:

(2) $$a_1 + b_1 + c_1 + d_1 = a_4 + b_4 + c_4 + d_4.$$

Adding equations (1) and (2) we get $a_1 + d_1 = b_4 + c_4$.

Having proved this relation, a number of further such symmetries can be anticipated. Thus, the relation $p + q = r + s$ will hold in each of the diagrams shown in Figure 4. The method of proof will be identical in all three cases.

	p	q	
r			s

	r	s	
p			q

p			
		s	
	r		
			q

Figure 4

Custom-made fourth-order magic squares! The relations uncovered above provide us with a way for constructing fourth-order magic squares in which the numbers in the top row have been filled in an arbitrary manner. For example, they could be the numbers which give your birth date. To show how this is done, we construct a magic square in which the top row has the numbers 15, 8, 19, 47 (these numbers codify an important date in Indian history).

15	8	19	47
a_2	b_2	c_2	d_2
a_3	b_3	c_3	d_3
a_4	b_4	c_4	d_4

The magic sum of the square is 15 + 8 + 19 + 47 = 89. There are as many as twelve unknowns in this array! The first question is: where do we start? We choose to start with the corner cells of the bottom row. We have:

$$a_4 + d_4 = 8 + 19 = 27.$$

Let us arbitrarily assign a pair of values to a_4, d_4, keeping the above condition in mind. Of course, we make sure that we do not use any of the numbers that have already occurred in the top row. Let us choose: $a_4 = 10$, $d_4 = 17$. (The choice is purely arbitrary.) We now update the array:

15	8	19	47
a_2	b_2	c_2	d_2
a_3	b_3	c_3	d_3
10	b_4	c_4	17

Next we choose to fill the cells in the central 2 × 2 block. We have:

$$b_2 + c_3 = 10 + 47 = 57,$$

$$b_3 + c_2 = 15 + 17 = 32.$$

We arbitrarily select: $b_2 = 5$, $c_3 = 52$, $b_3 = 28$, $c_2 = 4$. These yield: $b_4 = 89 - (8 + 5 + 28) = 48$, $c_4 = 89 - (19 + 4 + 52) = 14$. Once again, our choices must be such that no numbers are repeated. We now update the array:

15	8	19	47
a_2	5	4	d_2
a_3	28	52	d_3
10	48	14	17

Now write x for a_2. Then we have, since the magic sum is 89:

$$d_2 = 89 - 9 - x = 80 - x,\ a_3 = 89 - 25 - x = 64 - x,$$

$$d_3 = 89 - 64 - (80 - x) = x - 55.$$

As all the numbers are required to be positive, we must have the following:

$$x > 55,\ x < 64, \quad \therefore 56 \le x \le 63.$$

The numbers must also all be unequal, hence x, $80 - x$, $64 - x$, $x - 55$ must be different from all of the following:

15, 8, 19, 47, 5, 4, 28, 52, 10, 17, 48, 14.

Trying out the choices one by one, we find that $x = 57$ works. Here is the magic square it yields:

15	8	19	47
57	5	4	23
7	28	52	2
10	48	14	17

The reader may remark here that we have been lucky: no cases occurred of repeated numbers. Yes, we were lucky. But if indeed some repetition of numbers had happened, all that we would have to do is backtrack and make a different choice at some stage. In general, there are enough numbers available that we will obtain what we seek!

Exercises

Construct fourth-order magic squares in which the numbers in the first row are as given below:

1. (22, 12, 18, 87); this refers to Ramanujan's birthday (22nd December, 1887);
2. (14, 3, 18, 79); this refers to Albert Einstein's birthday (14th March, 1879);
3. Your own birthday!

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in the classroom

A magic square in Ramanujan's Honour

CoMaC

In recent months, a Power-Point presentation file on a fourth-order magic square has been doing the rounds on the internet. It is titled "Ramanujan's magic square" and it is written in a rather dramatic style. We give the gist of its content below, and then we ask you to account for the observed properties of the square using the theorems about fourth-order magic squares established elsewhere in this issue. (In some cases, we do not use the theorems themselves but extensions of those results. We ask you to prove the extensions for yourself.)

The square uses Ramanujan's birthday (December 22, 1887) to fill the cells in the top row (by now, you should know how to construct fourth-order magic squares with any given top row).

The author now asks: "What's so great in this?" and proceeds to answer the question himself or herself (as is usual with many such mails, one does not quite know whether the author is a man or a woman—or indeed who the author is at all!).

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

The first point that the author makes is that the row sums are all equal to 139, and so are all the column sums and the sums of the two main diagonals. (Please check for yourself that this is so.) This, of course, merely affirms that the structure is a magic square.

Keywords: *Ramanujan, magic square*

But now follow several other points of interest:

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

The sum of the four corner elements of the square is the same number (note the numbers in the cells coloured red), i.e., 22 + 87 + 11 + 19 = 139.

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

The sums of the numbers in the two sets of like-coloured cells are again the same number (139).

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

The sums of the numbers in the two sets of like-coloured cells here are yet again the same number (139).

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

The sum of the numbers in the four central cells is 139.

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

The sums of the numbers in these like-coloured 2 × 2 blocks are all 139.

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

And so also in these two coloured 2 × 2 blocks.

We invite the reader to relate all these observations with the theorems we have proved about fourth-order magic squares, and then to appreciate for himself or herself that every fourth-order magic square has these very same properties. And that is truly magical indeed.

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in the classroom

How To Prove It

In this article, we offer a second proof of the triangle-in-a-triangle theorem, using the principles of similarity geometry. Then, using vectors, we prove a result which is a generalisation of that theorem. We also give a pure geometry proof of the generalisation.

SHAILESH A SHIRALI

Given an arbitrary $\triangle ABC$ and a number t between 0 and 1, we locate points D, E, F on sides BC, CA, AB (see Figure 1) such that

$$\frac{BD}{BC} = \frac{CE}{CA} = \frac{AF}{AB} = t.$$

Segments AD, BE, CF when drawn intersect and demarcate a triangle PQR within the larger triangle ABC. The question now is: What is the ratio $f(t)$ of the area of $\triangle PQR$ to that of $\triangle ABC$? We showed in the previous issue that

$$f(t) = \frac{(2t-1)^2}{1-t+t^2}. \qquad (1)$$

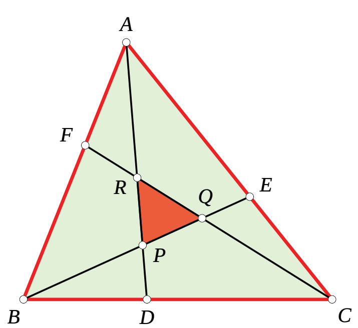

Figure 1.

Keywords: *Routh's theorem, triangle, ratio, area, similarity, vector*

In this issue, we shall modify the proof to obtain a generalisation of this result, known as *Routh's theorem*. But before we do that, we offer a 'pure geometry' proof for the formula for $f(t)$. We show that the derivation can be done by using the geometry of similar triangles. The derivation is due to **Swati Sircar** of Azim Premji University.

A Pure Geometry Proof

In general, a solution by pure geometry involves the construction of a few (appropriately and ingeniously chosen) auxiliary lines and circles. Here the steps we perform are the following: draw lines through B, C, A parallel respectively to segments AD, BE, CF. These lines intersect in pairs and create triangle XYZ (Figure 2). Extend segments AD, BE, CF to meet lines XY, YZ, ZX at points U, V, W respectively. Observe that in the resulting figure there are numerous triangles similar to $\triangle PQR$. We shall use these similarity relations to arrive at the desired answer.

Our strategy will be to first find the ratios $AR : RP : PU$, $BP : PQ : QV$ and $CQ : QR : RW$. We start by examining the ratios $BP : PQ : QV$. We already know the ratio $BQ : QV$, for by similarity,

$$\frac{BQ}{QV} = \frac{BF}{FA} = \frac{1-t}{t}.$$

As the ratio $(1-t)/t$ recurs all through the computation, it is convenient to have a symbol to denote it. Let $k = (1-t)/t$; then $BF/FA = AE/EC = CD/DB = k$, and also $BQ/QV = k$, $AP/PU = k$, $CR/RW = k$.

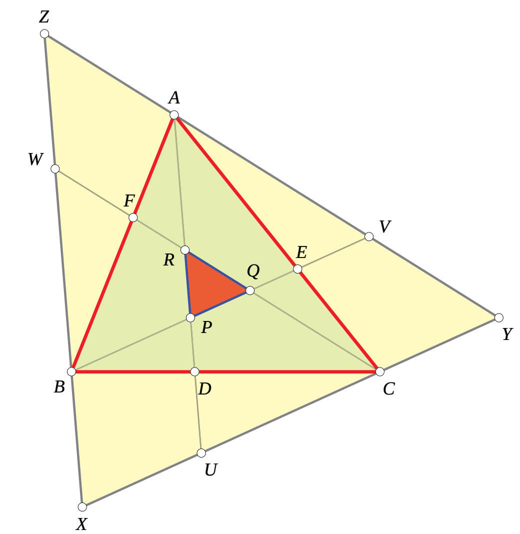

Figure 2.

Let $PQ/QV = \alpha$; we must determine α. We then have (the situation being schematically depicted as in Figure 3):

$$BP : PQ : QV = (k-\alpha) : \alpha : 1. \qquad (2)$$

By similarity we have $AR : RP = VQ : QP$. Since $AP : PU = k$, it follows that

$$AR : RP : PU = k : k\alpha : (\alpha + 1). \qquad (3)$$

Next, we have $CR : RW = k$ and

$$\frac{CQ}{QR} = \frac{UP}{PR} = \frac{1+\alpha}{k\alpha}.$$

Hence $CQ : RW = k : 1 + \alpha$, and:

$$CQ : QR : RW = k\big(1 + (k+1)\alpha\big) : k^2\alpha : 1 + (k+1)\alpha. \qquad (4)$$

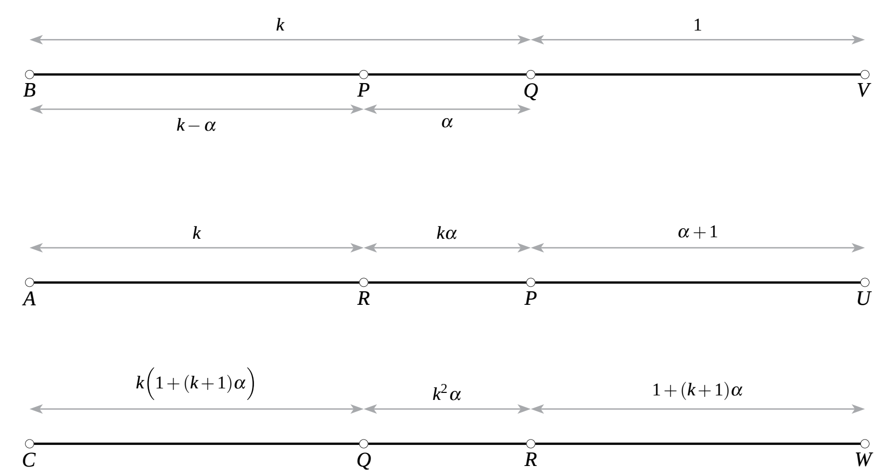

Figure 3.

Now we return to the original segment $BPQV$ (so the wheel has "turned full circle"). Since $BP : PQ = WR : RQ$, we have:

$$\frac{1+(k+1)\alpha}{k^2\alpha} = \frac{k-\alpha}{\alpha},$$
$$\therefore\ \alpha + (k+1)\alpha^2 = k^3\alpha - k^2\alpha^2,$$
$$\therefore\ \alpha = \frac{k^3-1}{k^2+k+1} = k-1,$$

a compact and pleasing result. On substituting this into the various expressions, we find that

$$BP : PQ : QV = AR : RP : PU = CQ : QR : RW = 1 : k-1 : 1 = t : 1-2t : t. \quad (5)$$

In particular we have $BP = QV$, $AR = PU$ and $CQ = RW$.

Next we determine the ratio $QE : QV$. We have:

$$\frac{QE}{EV} = \frac{EC}{AE} = \frac{t}{1-t}, \qquad \therefore\ \frac{QE}{QV} = t.$$

This implies that

$$BP : PQ : QE = t : 1-2t : t^2, \quad (6)$$

and it yields:

$$\frac{BQ}{BE} = \frac{1-t}{1-t+t^2}. \quad (7)$$

The ratios $AP : AD$ and $CR : CF$ are given by the same expression. The rest of the derivation proceeds as in the vector solution: the ratio of the sum of areas of $\triangle ABP$, $\triangle BCQ$ and $\triangle CAR$ to $\triangle ABC$ is

$$\frac{1-t}{1-t+t^2} \times 3t = \frac{3t(1-t)}{1-t+t^2},$$

hence the required ratio is 1 minus this quantity, i.e.:

$$\frac{\text{Area}\,(\triangle PQR)}{\text{Area}\,(\triangle ABC)} = 1 - \frac{3t(1-t)}{1-t+t^2} = \frac{(1-2t)^2}{1-t+t^2}. \quad (8)$$

Remark. Before proceeding, we pause to consider a special case of the above formula. Put $t = n/(2n+1)$; then we get, after simplification:

$$\frac{\text{Area}\,(\triangle PQR)}{\text{Area}\,(\triangle ABC)} = \frac{1}{1+3n+3n^2}.$$

This value of t corresponds to dividing the sides of the triangle into $2n+1$ equal parts (using $2n$ points equally spaced along the sides) and requiring that D, E, F lie at the points of division which are closest to the respective side midpoints. As you can see, it results in a simple formula for the areal ratio. The choice $2n+1 = 3$ (which comes from $n = 1$) corresponds to trisecting the sides; it results in the areal ratio $1 : 7$.

The denominator in the above formula, $3n^2+3n+1$, generates the following sequence (by putting $n = 0, 1, 2, \ldots$):

$$1,\ 7,\ 19,\ 37,\ 61,\ 91,\ \ldots.$$

These numbers are the differences between consecutive cubes: $1 = 1^3 - 0^3$, $7 = 2^3 - 1^3$, $19 = 3^3 - 2^3$, and so on. To see why this is true in general, note the following simple identity: $3n^2+3n+1 = (n+1)^3 - n^3$. The very same numbers are generated by the sequence of centred hexagonal dot figures shown in Figure 4. This explains why they are sometimes called the *centred hexagonal numbers*.

Routh's Theorem

Now we consider a more somewhat general configuration. We start as earlier with an arbitrary $\triangle ABC$, but now we have three numbers u, v, w (rather than just one number t), all between 0 and 1. We now locate points D, E, F on sides BC, CA, AB (see Figure 5) such that

$$\frac{BD}{BC} = u, \quad \frac{CE}{CA} = v, \quad \frac{AF}{AB} = w.$$

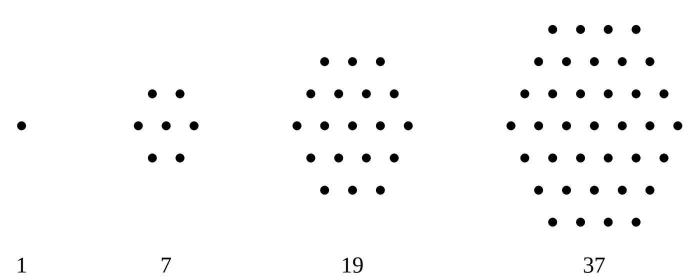

Figure 4.

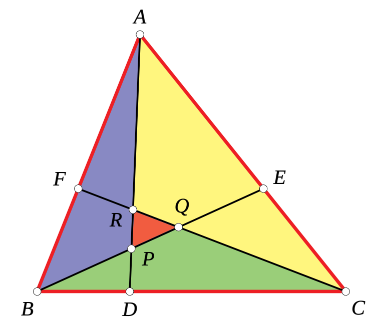

Figure 5.

As earlier, segments AD, BE, CF when drawn intersect and demarcate a triangle PQR within the larger triangle ABC. The question now is: What is the ratio $g(u,v,w)$ of the area of $\triangle PQR$ to that of $\triangle ABC$?

Let B be treated as the origin, and let

$$\overrightarrow{BC} = \mathbf{c}, \qquad \overrightarrow{BA} = \mathbf{a}.$$

By construction we have

$$\overrightarrow{BD} = u\,\overrightarrow{BC} = u\mathbf{c}, \quad \overrightarrow{CE} = v\,\overrightarrow{CA} = v\,(\mathbf{a}-\mathbf{c}),$$
$$\overrightarrow{AF} = w\,\overrightarrow{AB} = -w\mathbf{a}.$$

Let $AP/AD = k$. To find the unknown quantity k, we argue as follows.

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = -\mathbf{a} + u\mathbf{c},$$
$$\therefore\ \overrightarrow{AP} = k\,\overrightarrow{AD} = -k\mathbf{a} + ku\mathbf{c},$$
$$\therefore\ \overrightarrow{BP} = \overrightarrow{BA} + \overrightarrow{AP} = (1-k)\mathbf{a} + ku\mathbf{c}.$$

We also have, by similar logic or by using the section formula:

$$\overrightarrow{BE} = v\mathbf{a} + (1-v)\mathbf{c}.$$

Now consider the last two results we have obtained:

$$\overrightarrow{BP} = (1-k)\mathbf{a} + ku\mathbf{c}, \quad (9)$$
$$\overrightarrow{BE} = v\mathbf{a} + (1-v)\mathbf{c}. \quad (10)$$

Vectors $\overrightarrow{BP}$ and $\overrightarrow{BE}$ are parallel and have been expressed in terms of the non-zero, non-parallel vectors **a** and **c**. Hence **a** and **c** must be mixed in the same proportions in $\overrightarrow{BP}$ and $\overrightarrow{BE}$, and we have:

$$\frac{1-k}{ku} = \frac{v}{1-v}. \quad (11)$$

This allows us to find the unknown quantity k:

$$\frac{AP}{AD} = \frac{1-v}{1-v+uv}. \quad (12)$$

In the same way (or by using symmetry), we find the ratios $BQ : BE$ and $CR : CF$:

$$\frac{BQ}{BE} = \frac{1-w}{1-w+vw},$$
$$\frac{CR}{CF} = \frac{1-u}{1-u+wu}.$$

Having found these ratios, we see next that

$$\frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ABD} = \frac{1-v}{1-v+uv}.$$

We also know that $BD/BC = u$. From this it follows that:

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ABC} = u.$$

Hence by multiplication we get:

$$\frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ABC} = \frac{u(1-v)}{1-v+uv}.$$

In the same way (or by using symmetry), we find that:

$$\frac{\text{Area of } \triangle BCQ}{\text{Area of } \triangle ABC} = \frac{v(1-w)}{1-w+vw},$$
$$\frac{\text{Area of } \triangle CAR}{\text{Area of } \triangle ABC} = \frac{w(1-u)}{1-u+wu}.$$

Hence we have:

$$\frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle ABC} = 1 - \frac{u(1-v)}{1-v+uv} - \frac{v(1-w)}{1-w+vw} - \frac{w(1-u)}{1-u+wu}.$$

This yields the desired formula

$$g(u,v,w) = 1 - \frac{u(1-v)}{1-v+uv} - \frac{v(1-w)}{1-w+vw} - \frac{w(1-u)}{1-u+wu}. \quad (13)$$

This result is generally known as Routh's theorem, named after Edward John Routh who first mentioned it in a book published in 1896. (However, it appears to have been known much before that date. It was used as an examination question in the famous Mathematical Tripos.)

We may verify after algebraic manipulation that $g(t,t,t) = f(t)$. If we define a new set of quantities x, y, z by:

$$x = \frac{BD}{DC}, \quad y = \frac{CE}{EA}, \quad z = \frac{AF}{FB},$$

then the statement of the result assumes a slightly more convenient form:

$$\frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle ABC} = \frac{(xyz-1)^2}{(xy+x+1)(yz+y+1)(zx+z+1)}. \quad (14)$$

Pure geometry proof of Routh's theorem. Pure geometry proofs of the theorem have been known for some time. We conclude this article with one such proof which appeared in the magazine *Crux Mathematicorum*; see [2]. The proof is due to James S. Kline and Daniel J. Velleman, and is both ingenious and compact.

In Figure 5, we have drawn lines through P which are parallel to the sides of the triangle, thus giving rise to segments $GH \parallel BC$, $IJ \parallel CA$ and $KL \parallel AB$. (Note: we have suppressed the labels of points F, R, Q to avoid a visual clutter.) Our objective is to find the ratio of the area of $\triangle ABP$ to the area of $\triangle ABC$. Towards this end, we shall find the ratio JP/AC. We shall work in terms of x, y, z rather than u, v, w. (Recall that $BD/DC = x$, $CE/EA = y$ and $AF/FB = z$.) By triangle similarity, we have:

$$\frac{JP}{PI} = \frac{AE}{EC} = \frac{1}{y},$$

hence $PI = y \cdot JP$.

Again, $\triangle PJG \sim \triangle HKP$ and $\triangle AGH \sim \triangle ABC$, hence:

$$\frac{HK}{PJ} = \frac{HP}{PG} = \frac{CD}{DB} = \frac{1}{x},$$

hence $HK = JP/x$.

The expressions for the lengths of PI and HK now lead to the following:

$$\begin{aligned} CA &= CH + HK + KA \\ &= IP + HK + PJ \\ &= y \cdot JP + \frac{JP}{x} + JP. \end{aligned}$$

Hence we have:

$$\frac{JP}{AC} = \frac{1}{1 + 1/x + y} = \frac{x}{xy+x+1}.$$

The ratio of the altitudes of $\triangle ABP$ and $\triangle ABC$ through P and C respectively must be given by the

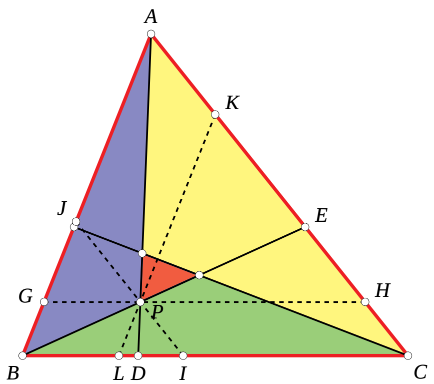

Figure 6.

same expression, $x/(xy+x+1)$. As the two triangles share the same base AB, the ratio of the area of $\triangle ABP$ to that of $\triangle ABC$ is given by that very same expression:

$$\frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ABC} = \frac{x}{xy+x+1}.$$

We derived this expression by drawing lines through P which are parallel to the sides of $\triangle ABC$. By similarly drawing lines through Q and R respectively which are parallel to the sides of the triangle (we have not shown these lines), we derive expressions for the ratios of the areas of $\triangle BCQ$ and $\triangle CAR$ to that of $\triangle ABC$. We obtain the following:

$$\frac{\text{Area of } \triangle BCQ}{\text{Area of } \triangle ABC} = \frac{y}{yz+y+1},$$

$$\frac{\text{Area of } \triangle CAR}{\text{Area of } \triangle ABC} = \frac{z}{zx+z+1}.$$

Hence:

$$\begin{aligned} \frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle ABC} &= 1 - \frac{x}{xy+x+1} \\ &\quad - \frac{y}{yz+y+1} - \frac{z}{zx+z+1} \\ &= \frac{(xyz-1)^2}{(xy+x+1)(yz+y+1)(zx+z+1)} \end{aligned}$$

The last step requires a bit of algebraic jugglery, which we leave you to undertake. For more on Routh's theorem, see [1] and [3].

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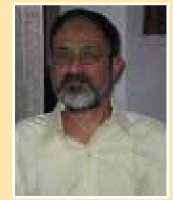
1. M. S. Klamkin and A. Liu, "Three More Proofs of Routh's Theorem." *Crux Mathematicorum* **7**, 199-203, 1981
2. J. S. Kline and D. Velleman. "Yet another proof of Routh's theorem," *Crux Mathematicorum* **21**, 37-40, 1995
3. Routh's theorem, Wikipedia. https://en.wikipedia.org/wiki/Routh's_theorem



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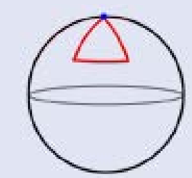
SOUTH-EAST-NORTH PUZZLE

- SOURAV SEN GUPTA

Assume that the world is a perfect three-dimensional sphere, and you are standing at a specific point on the surface of the world. Suppose you travel 100 km to the South, then 100 km to the East, and finally 100 km to the North, and you find yourself back at the starting point of your journey. At which point on the surface did you start? (This puzzle appeared as a problem in the B Stat/B Math entrance test of Indian Statistical Institute.)

TRIVIAL SOLUTION

If you have seen this puzzle before, or if you have pondered over it for some time, you must have arrived at the trivial solution, that is, the North Pole! You are correct – if you start from the North Pole and travel as prescribed, your path will look like a 'triangle' on the surface of the sphere as shown in the figure at the left.



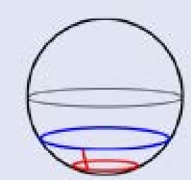
EXTENDED VERSION

Now let's make the question more interesting. Are there any other points, apart from the North Pole, which satisfy the same conditions?

You may be surprised to know that there are infinitely many non-trivial solution points to this puzzle, apart from the obvious one – the North Pole. The problem with our generic thought pattern is that we visualize a 'triangle' as soon as we see the words "to the South, then to the East, and then to the North". What we overlook is that one may wrap around the world in a circular path if one continues moving East. Consider the following.

NON-TRIVIAL SOLUTIONS

Suppose that the circumference of the red circle shown in the second figure is exactly 100 km (you can prove that there must exist such a circle), and the red circle is 100 km south of the blue one. Then, starting from *any* point on the blue circle, travelling 100 km South brings you to a point on the red circle, 100 km East wraps around the red circle and brings you back to the same point, and then 100 km North brings you back to the starting point on the blue circle. As there are infinitely many points on the blue circle, you have infinitely many solutions to this puzzle!



TAILPIECE

We have just pointed out that the conditions of the puzzle allow infinitely many solutions other than the North Pole. The reader may be surprised to hear that in addition to this infinite set, there are yet more solutions! But we shall leave it to you to identify these additional solutions.

in the classroom

The Cost of Money

COMAC

In a post that appeared early in 2014 on the site http://mathforlove.com/2014/02/a-dollar-that-costs-a-dollar/, a very interesting and mathematically elegant question was posed which we study here. Consider the US coinage system (coins only; we do not include any notes of value $1 or more). We have the following coins: Penny (1 cent, $ 0.01), Nickel (5 cents, $ 0.05), Dime (10 cents, $ 0.10) and Quarter (25 cents, $ 0.25). Each of these coins is made of some metal or mixture of metals, and as such each one has a cost associated with it. After all, the metals in question have to be purchased from the market! At the time the post was written, here were the relevant costs (the cost is given in cents per coin):

Coin	Penny	Nickel	Dime	Quarter
Market cost (cents)	2.5	11	6	11

The author asked the following questions of his students:
(a) Which mix of coins makes the cheapest dollar? (b) Which mix of coins makes the most expensive dollar? These questions are easy to answer and we do not study them. But a question was posed by one of the girls in the class which was mathematically far more rich:

Keywords: *Money, value, cost, multiple, factor, remainder, modulus*

Can you make a dollar in coins that also ***costs*** *a dollar to make?*

Let's see what this questions involves. To make the dollar using only dimes, we need ten dimes; that would cost 60 cents. Not good enough. How about a dollar made up of twenty nickels? That costs 20 × 11 = 220 cents. No, that does not work either. How about eight dimes, two nickels, and ten pennies? That costs 95 cents. Close, but still not good enough! Can we find a combination whose cost is exactly $1?

The author reports that the girl found a solution after an intense session lasting thirty-five minutes. He marvels not just at her creativity in asking such a rich question but also at her diligence in finding an answer: (He writes: "I had one of those awesome experiences this week where a student thinks of a better question." Would that we all have more such experiences!)

We shall address the same question here. But being math teachers we shall go a step further: we shall ask for *all* possible solutions.

Mathematical formulation. Let a, b, c, d denote the numbers of pennies, nickels, dimes and quarters used. Then the monetary value of this 'portfolio' is $a + 5b + 10c + 25d$ cents, and the market cost is $2.5a + 11b + 6c + 11d$ cents (using the cost data given above). Hence we must find non-negative integers a, b, c, d which satisfy the following system:

(1) $a + 5b + 10c + 25d = 100,$

(2) $2.5a + 11b + 6c + 11d = 100.$

Our task is to find all the solutions to this system.

Equation (1) tells us that a is a multiple of 5 (because $5b$, $10c$, $25d$ and 100 are all multiples of 5), and equation (2) tells us that a is even. Hence a is a multiple of 10. Let $a = 10x$ where x is a non-negative integer. Substituting $a = 10x$ in both the equations and simplifying, we get:

(3) $2x + b + 2c + 5d = 20,$

(4) $25x + 11b + 6c + 11d = 100.$

From equation (3) we get $b = 20 - 2x - 2c - 5d$. Substituting for b in equation (4) we get

$25x + 11(20 - 2x - 2c - 5d) + 6c + 11d = 100,$ and therefore

(5) $3x - 16c - 44d + 120 = 0.$

From equation (5) we see that $3x$ is a multiple of 4 and hence that x is a multiple of 4. Let $x = 4y$ where y is a non-negative integer. (So the relation between a and y is: $a = 40y$.)

The system now becomes:

(6) $8y + b + 2c + 5d = 20,$

(7) $100y + 11b + 6c + 11d = 100.$

From equation (7) we see that y cannot exceed 1. Thus, y can only be 0 or 1.

If $y = 1$ then equation (7) implies that $b = 0$, $c = 0$, $d = 0$. But then equation (6) is not satisfied. So this possibility does not lead to a solution. Hence $y = 0$ (which means that $x = 0$ and hence also $a = 0$). This yields:

(8) $b + 2c + 5d = 20,$

(9) $11b + 6c + 11d = 100.$

We now make use of the terms $11b$ and $11d$ in equation (9); they are (obviously) multiples of 11. Since $100 \equiv 1 \pmod{11}$, it follows that $6c \equiv 1 \pmod{11}$, and hence that c is one of the following numbers: 2, 13, 24, 35, ... Also from equation (8) we get $2c \le 20$, i.e., $c \le 10$. Hence $c = 2$. Therefore we get:

$$b + 5d = 16,$$

$$b + d = 8.$$

This pair of equations may be solved to yield $b = 6$, $d = 2$. So we obtain a solution:

$$(a, b, c, d) = (0, 6, 2, 2).$$

Please check that these values do satisfy the original equations. *This is the only solution possible*. That is, the only answer to the proposed problem is:

- 0 pennies;
- 6 nickels (worth $0.30 and costing $0.66);
- 2 dimes (worth $0.20 and costing $0.12);
- 2 quarters (worth $0.50 and costing $0.22).

References

1. http://mathforlove.com/2014/02/a-dollar-that-costs-a-dollar/
2. http://mathforlove.com/wp-content/uploads/2014/02/ How-Much-Does-Money-Cost-v2-simplification.pdf

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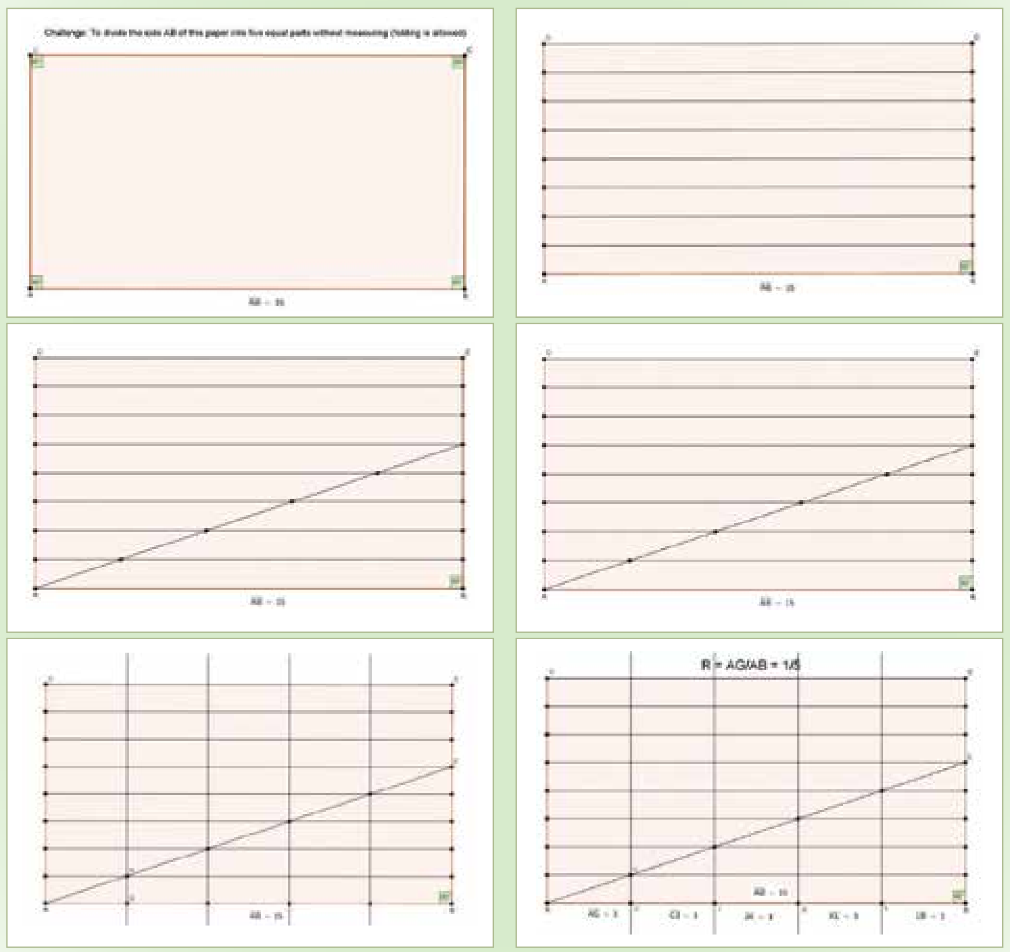
A GRAPHIC MATH STORY



1. What has been done?
2. Why did it work?
3. Will it work for any measure of AB?

in the classroom

Investigations

Tanuj Shah

Investigations are an integral part of the Maths curriculum in the UK, and rubrics have been created to assess children's work in this area. In India, however, investigations have not yet become a part of the maths curriculum, though the activity lends itself in a natural way to interesting mathematical work; in some ways it simulates the way a mathematician works.

An investigation can have a seemingly simple looking problem as its starting point but can lead to lines of inquiry which provide rich insight into a particular area of mathematics. It is important to let children develop their own lines of inquiry, and to have the experience of encountering 'dead ends.' In particular, one must not lead their inquiry but provide broad pointers for developing further lines of inquiry.

In this issue of *At Right Angles* we will look at an investigation which will help children 'discover' an important mathematical relationship which is generally introduced in class 7. Ideally, an hour should be allocated for this investigation, so that even the slowest student in the class is able to complete it, while at the same time there are extensions to this task which can be pursued by the quicker ones. If it is difficult to find a one hour slot, then three quarters of an hour would be sufficient, with the teacher having to provide more hints to the students.

Keywords: *Pattern, arithmetic, algebra, identities, squares, difference, sum, investigation*

Start by posing this question to the students:

Work out

$$9^2 - 8^2.$$

Is there an easier way of computing the above, without having to square the numbers?

If this is the first time the children are doing an investigation, then you could elicit from the students how to move forward with this problem. Some suggestions along the lines of working out a few more problems in the standard way and looking for patterns may come from the students (or the teacher may have to suggest it). You may then let the children get on with the problem, instructing them that when they find something they should write down their answer and then raise their hand, so you can go and check their solution. At no point should they shout out any thoughts they may have regarding the problem (to avoid disrupting other students' lines of inquiry).

You may want to orient the children in terms of the attitudes and qualities required when doing an investigation. They would need patience, and they must not be in a hurry to find the answer. In fact, there may be more than one answer; sometimes they may discover patterns which even the teacher may not have seen. Sometimes, they will encounter 'dead ends,' so the need to persevere should be emphasised. They should also get the sense that there is sufficient time to tackle the investigation and, therefore, that they are not working under any time pressure.

You will find that the children adopt a wide variety of approaches in tackling an investigation. Some children are well organized and systematic in their approach, and will have looked only at square numbers which are consecutive; they will soon be able to come up with the observation that adding the two numbers is an easier computation. They could be then asked to extend this investigation to square numbers which are not consecutive. Meanwhile, you will also see that some children are not systematic in their approach and therefore unable to see any patterns. They could be asked to look at the example again and see the relationship between the two numbers. Eventually, for those not able to get it even after the hints that you have given, you could tell them that the numbers in the example are consecutive numbers.

Student 1	**Student 2**
$4^2 - 3^2 = 7$	$5^2 - 3^2 = 16$
$7^2 - 6^2 = 13$	$8^2 - 7^2 = 15$
$10^2 - 9^2 = 19$	$10^2 - 8^2 = 36$

Children who would have seen the need to be systematic would soon be trying examples where the gap between the two numbers is 1, then 2, then 3, etc. It will not be long before they will begin to see that if the gap is 1, then you need to add the two numbers; if the gap is 2, then you need to add the two numbers and double your answer, and if the gap is 3, you need to add the two numbers and multiply by 3. They should then be encouraged to write down such statements in algebraic form. You may begin to get generalisations like this:

$$x^2 - y^2 = (x + y)\, g,$$

where g is the gap between the two numbers. A little bit of prodding to make them think about how the gap can be represented algebraically will lead them to the identity:

$$x^2 - y^2 = (x + y)(x - y).$$

Those who are quick at coming upon the identity could be encouraged to extend the investigation by studying $5^3 - 4^3$, just to see if they could come up with a way of getting the answer in which they do not need to cube the numbers. You could give them a hint that squares of the numbers may help here.

Sometimes, children come up with interesting discoveries which we ourselves may not have seen earlier. For example, in studying the differences between consecutive cube numbers, children may come up with the following relation: $x^3 - y^3 = 3xy + 1$, where x and y are consecutive numbers. Others may come up with the more standard form $x^3 - y^3 = x^2 + x\,y + y^2$. You could ask the students to show how these two expressions which look very different are actually the same. Once a certain buzz has been created, you will find children attempting to look at differences of higher powers. Looking at fourth powers would be well within their reach.

One of the things about an investigative activity is that it is accessible for ALL the students in a class and the teacher can use his or her judgment on the level of support to be given to different individuals. For example, the teacher would tell the slower students to choose smaller numbers when generating their own examples. When children 'discover' the identity for the difference between two squares, they have an ownership of it and thus a greater chance of retention and an ability to use it in different situations.

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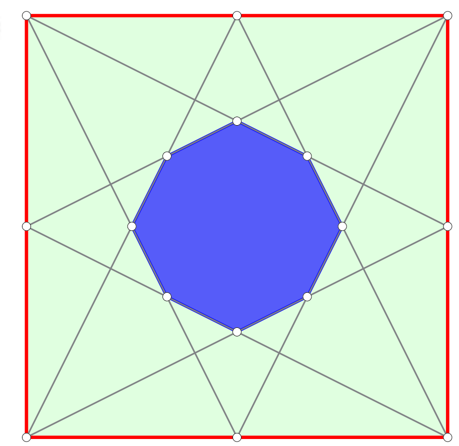
OCTAGON IN A SQUARE



The picture above shows a square in which line segments have been drawn from each vertex of the square to the midpoints of the two sides remote from that vertex (i.e., the sides which do not touch that vertex). Eight line segments have thus been drawn within the square, creating an octagon, shown in blue. Here are two questions for you relating to this octagon.

1. Is the octagon regular? (Recall that a polygon is said to be regular if its sides haveequal length and its internal angles have equal measure.) Prove your answer!
2. What is the ratio of the area of the octagon to that of the square?

E-mail your answers to AtRiA.editor@apu.edu.in.

techspace

Hill Ciphers

Jonaki B Ghosh

Introduction

Cryptography is the science of making and breaking codes. It is the practice and study of techniques for secure communication. Modern cryptography intersects the disciplines of mathematics, computer science, and electrical engineering. Applications of cryptography include ATM cards, computer passwords, and electronic commerce.

This article is a sequel to an article which appeared in the November 2014 issue of *At Right Angles* in which we had described an interesting cryptography method known as the *Hill Cipher*. (See http://www.teachersofindia.org/en/article/hill-ciphers.) The Hill Cipher method is based on matrices and modular arithmetic. As in the previous article, we will explore the method using the spreadsheet MS Excel in which we will perform operations on matrices.

Keywords: *Cryptography, cipher, matrix, augmented, inverse, identity, multiplication, transformation, plaintext, encoding, decoding, modular*

Hill Ciphers

We had described in the previous article that Hill ciphers are an application of matrices to cryptography. Ciphers are methods for transforming a given message, the *plaintext*, into a new form that is unintelligible to anyone who does not know the key (the transformation used to convert the plaintext). In a cipher the key transforms the plaintext letters to other characters known as the *ciphertext*. The secret rule, that is, the inverse key, is required to reverse the transformation in order to recover the original message. To use the key to transform plaintext into ciphertext is to *encipher* the plaintext. To use the inverse key to transform the ciphertext back into plaintext is to *decipher* the ciphertext.

In order to understand Hill ciphers, we need to understand modular arithmetic, and be able to multiply and invert matrices. We would urge the reader to refer to the article in the November 2014 issue. However, we shall mention some important definitions here.

Definition 1: An arbitrary Hill n-cipher has as its key a given $n \times n$ invertible matrix whose entries are non-negative integers from among $0, 1, \ldots, m-1$, where m is the number of characters used for the encoding process. Suppose we wish to use all the 26 alphabets from A to Z and three more characters, say '.', '–' and '?'. This means we will have 29 characters with which we can write our plaintext. These have been shown in the given substitution table where the 29 characters have been numbered from 0 to 28.

Let us recall the encryption method by applying this to an example of a Hill 2-cipher corresponding to the substitution table (Table 1) with 29 characters. Let the key be the invertible 2×2 matrix

$$\mathrm{E} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

We can also refer to E as the 'encoding matrix'. We will use E to encipher groups of two consecutive characters. Suppose we have to encipher the word **HI**. The alphabets H and I correspond to the numbers 7 and 8, respectively, from the above substitution table. We shall represent it as a 2×1 matrix.

$$\begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

To encipher **HI**, we shall pre-multiply this matrix by the encoding matrix E.

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 38 \\ 68 \end{bmatrix}$$

The product is a 2×1 matrix with entries 38 and 68. But what characters do the numbers 38 and 68 represent? These are not in our substitution table! What we shall do is as follows:

We will divide these numbers by 29 and consider their respective remainders after the division process is done. Thus when we divide 38 by 29, the remainder is 9 and when we divide 68 by 29, the remainder is 10.

To express this in the language of *modular arithmetic*, we write

$$38 \equiv 9 \pmod{29} \text{ and } 68 \equiv 10 \pmod{29}$$

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**
0	1	2	3	4	5	6	7	8	9	10	11	12
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
13	14	15	16	17	18	19	20	21	22	23	24	25
.	**—**	**?**										
26	27	28										

Table 1. The substitution table for the Hill Cipher

Definition 2: Given an integer $m > 1$, called the *modulus*, we say that the two integers a and b are *congruent* to one another *modulo m* and we write

$$a \equiv b \,(\text{mod } m) \quad \text{(we read this as '}a\text{ is congruent to }b\text{ modulo }m\text{')}$$

This means that the difference $a - b$ is an integral multiple of m. In other words, $a \equiv b \pmod{m}$ when $a = b + km$ for some integer k (positive, negative or zero)

For our Hill 2-cipher, we have

$$38 \equiv 9 \pmod{29} \text{ and } 68 \equiv 10 \pmod{29}$$

Note that 38 = 9 + 1 × 29 and 68 = 10 + 2 × 29.

The numbers 9 and 10 correspond to the alphabets J and K respectively from our substitution table.

Thus the word **HI** is enciphered to **JK**!

In order to use this method of sending secret messages, the sender has to encrypt the plaintext (the original message) **HI** and send the encrypted form **JK** to the receiver. The secret key, that is, the encoding matrix E is known only to the sender and the receiver. Now let us see how the receiver can decipher what **JK** stands for.

In order to decipher **JK**, we begin by looking for the numbers corresponding to **J** and **K** in our substitution table. These are 9 and 10 respectively. We represent this in the form of a 2 × 1 matrix

$$\begin{bmatrix} 9 \\ 10 \end{bmatrix}$$

In the previous article we had used the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$ as our encoding matrix. Note that we have to use the inverse of the encoding matrix to decipher the ciphertext. Thus we had used its inverse $\begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$ to decrypt the ciphertext. Note that $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$ has a determinant equal to 1. Thus the decryption was simple as we needed to multiply the inverse matrix with the message matrix (refer to pages 67-68 in the November 2014 issue). However this method is likely to pose a difficulty if the determinant of the encoding matrix is any value other than 1. This means that the inverse matrix, that is, the inverse key will comprise fractional entries. How will we then decode the encrypted text or ciphertext?

Thus, if the determinant of the encoding matrix is not equal to 1, as in the case of $E = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ (the determinant is equal to -2), we will need to find the inverse of the matrix in Z_{29}, the set of integers modulo 29. Note that the actual inverse is $E^{-1} = \frac{1}{-2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$ but this will not be helpful as two entries in the matrix are fractions. However, when we find the inverse in Z_{29}, all entries will be integers from 0 to 28 which will certainly serve our purpose. Note that the set $\{0, 1, 2, 3, \ldots, n-1\}$ is referred to as the *set of integers modulo n* and is represented as Z_n.

To find the inverse of a matrix in Z_{29}:

We shall now demonstrate the method of finding the inverse of a 2 × 2 matrix in $Z_{29.}$ First we shall augment the matrix E with the 2 × 2 identity matrix, I, to its right, and obtain $[E|\,I]$. Further we shall apply elementary row operations till we obtain $[I|\,E^{-1}]$.

Now,

$$[E|\,I] = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right]$$

Our aim is to convert E to I, using elementary row transformations. At the end of the process, I will be automatically converted to E^{-1}. In general we will perform row operations so that 2 gets converted to 1, 4 to 0, 5 to 1 and 3 to 0 (in that order). We shall refer to the first row of the augmented matrix as R_1 and the second row as R_2.

To begin the process we need to find the multiplicative inverse of 2 in Z_{29}. We shall thus multiply R_1 by 15 since 15 is the multiplicative inverse of 2 in Z_{29}. Note that $2 \times 15 = 30 \equiv 1 \pmod{29}$. In Table 2 we have included the multiplicative inverses of all integers in Z_{29} (in blue). The reader is urged to verify these values and use them as a reference for the remaining calculations.

Thus, performing the row operation $R_1 \to 15\,R_1$ on $[E|\,I]$ and reducing it modulo 29, we get

$$\left[\begin{array}{cc|cc} 30 & 45 & 15 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \approx \left[\begin{array}{cc|cc} 1 & 16 & 15 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \pmod{29}$$

Note that 2 in the original augmented matrix has been replaced by 1.

In order to convert 4 to 0 (in the reduced matrix), we need to perform the row operation $R_2 \to R_2 - 4 \times R_1$. This gives

$$\left[\begin{array}{cc|cc} 1 & 16 & 15 & 0 \\ 0 & -59 & -60 & 1 \end{array}\right] \approx \left[\begin{array}{cc|cc} 1 & 16 & 15 & 0 \\ 0 & 28 & 27 & 1 \end{array}\right] \pmod{29}$$

Now, to convert 28 to 1, we need to multiply 28 by its inverse in Z_{29}, that is, perform the row operation $R_2 \to 28\,R_2$. Note that 28 is its own inverse in Z_{29}, as $28 \times 28 = 784 \equiv 1 \pmod{29}$.We now get:

$$\left[\begin{array}{cc|cc} 1 & 16 & 15 & 0 \\ 0 & 784 & 756 & 28 \end{array}\right] \approx \left[\begin{array}{cc|cc} 1 & 16 & 15 & 0 \\ 0 & 1 & 2 & 28 \end{array}\right] \pmod{29}$$

Finally, to convert 16 to 0, we need to perform the row operation $R_1 \to R_1 - 16 \times R_2$. This gives us:

$$\left[\begin{array}{cc|cc} 1 & 0 & -17 & -448 \\ 0 & 1 & 2 & 28 \end{array}\right] \approx \left[\begin{array}{cc|cc} 1 & 0 & 12 & 16 \\ 0 & 1 & 2 & 28 \end{array}\right] \pmod{29}$$

We have now succeeded in converting the segmented matrix $[E|\,I]$ to $[I|\,E^{-1}]$.

Thus, the inverse of E = $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ in Z_{29} is $E^{-1} = \begin{bmatrix} 12 & 16 \\ 2 & 28 \end{bmatrix}$.

Now, to decrypt **JK** we need to perform the multiplication

$$\begin{bmatrix} 12 & 16 \\ 2 & 28 \end{bmatrix}\begin{bmatrix} 9 \\ 10 \end{bmatrix} \text{ and reduce it modulo 29.}$$

Thus,

$$\begin{bmatrix} 12 & 16 \\ 2 & 28 \end{bmatrix}\begin{bmatrix} 9 \\ 10 \end{bmatrix} = \begin{bmatrix} 268 \\ 298 \end{bmatrix} \approx \begin{bmatrix} 7 \\ 8 \end{bmatrix} \pmod{29}$$

7 and 8 can be traced back to the alphabets H and I and hence the plaintext message **HI**!

In the previous article, we had encrypted the plaintext **MATH_IS_FUN.** using the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$. Let us now encrypt the same using the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	15	10	22	6	5	25	11	3	3	8	17	9	27
15	16	17	18	19	20	21	22	23	24	25	26	27	28
2	20	12	21	26	16	18	4	24	23	7	19	14	28

Table 2.

The steps are indicated below. For matrix computations we use MS Excel. In Excel, the commands for multiplying matrices and finding the inverse of a matrix are MMULT and MINVERSE respectively. For reducing a number modulo a divisor the required command is MOD.

Encoding or enciphering the plaintext: The steps

Step 1: Convert the plaintext **MATH_IS_FUN.** to the corresponding substitution values from the substitution table. The values are

12 0 19 7 27 8 18 27 5 20 13 26

We need to make a $2 \times n$ matrix using these values

Step 2: Form pairs of these numbers as follows

12 0   19 7   27 8   18 27   5 20   13 26

Each pair will form a column of a 2×6 matrix (since there are 6 pairs). Let us call this matrix P (the plaintext matrix)

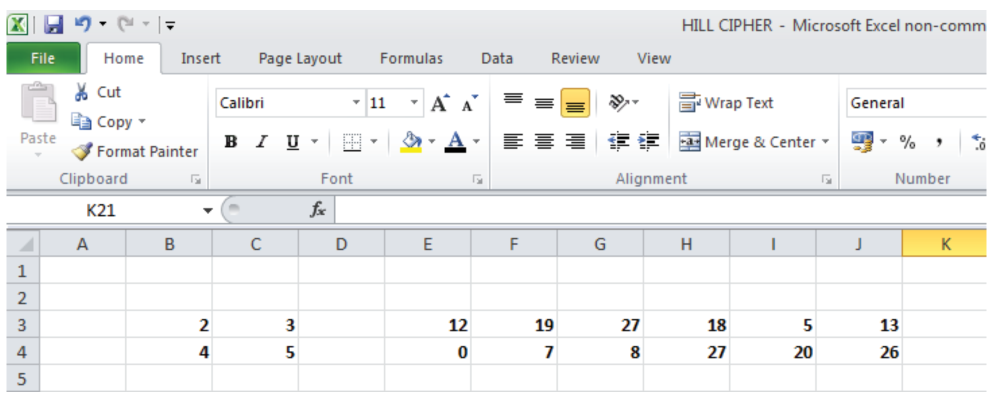
$$P = \begin{bmatrix} 12 & 19 & 27 & 18 & 5 & 13 \\ 0 & 7 & 8 & 27 & 20 & 26 \end{bmatrix}$$

Step 3: Compute the product EP

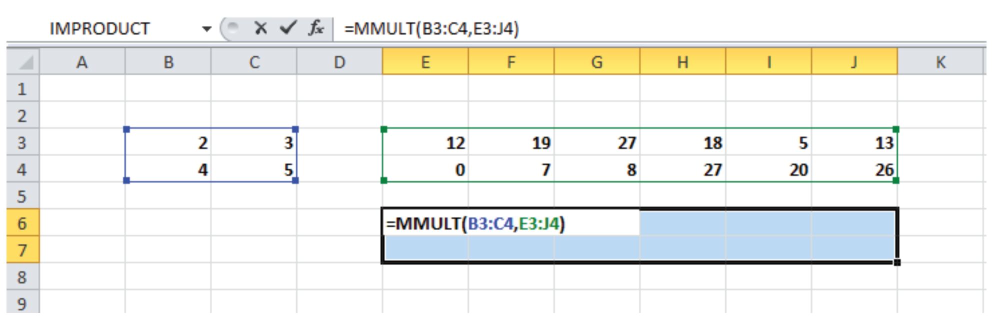
$$EP = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 12 & 19 & 27 & 18 & 5 & 13 \\ 0 & 7 & 8 & 27 & 20 & 26 \end{bmatrix} = \begin{bmatrix} 24 & 59 & 78 & 117 & 70 & 101 \\ 48 & 111 & 148 & 207 & 120 & 177 \end{bmatrix}$$

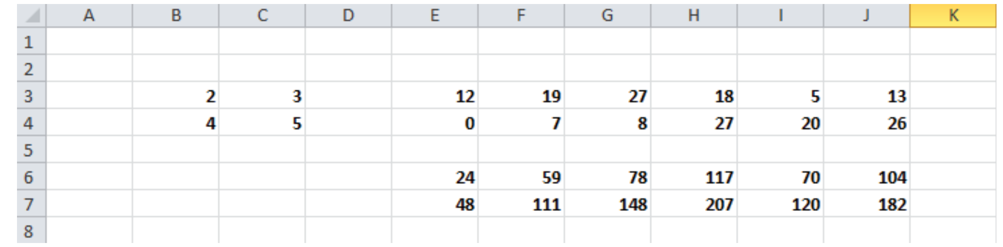
In order to perform this computation in Excel, we proceed as follows:

Enter the 2×2 matrix E and the 2×6 matrix P as separate arrays as shown. Each entry of a matrix may be entered by typing a number in a cell and pressing Enter. The arrow keys may be used to move to the next appropriate cell.



To obtain the product, select a blank 2×6 array and type **=MMULT(** in the top leftmost cell of the chosen array. Within the parentheses, first select the array for matrix E and then the array for matrix P separated by a comma. Press **Crtl + Shift** followed by **Enter** to obtain the product. (Note that you need to press **Crtl** and **Shift** simultaneously and then press **Enter**.)



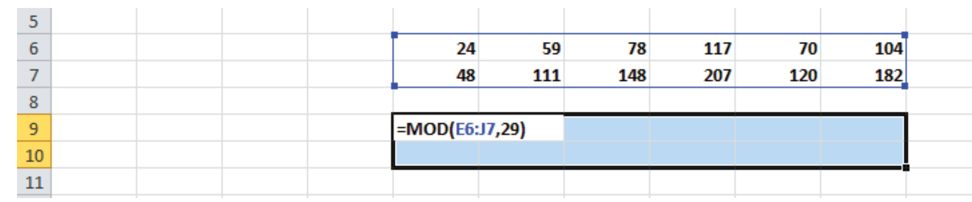


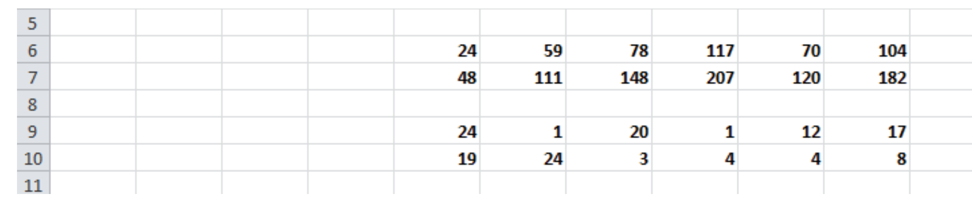
Step 4: Reduce the product modulo 29 to obtain the Hill 2-cipher values. This means we have to divide each number by 29 and find the remainder. In Excel we can reduce the entire matrix modulo 29 in one go!

$$EP = \begin{bmatrix} 24 & 59 & 78 & 117 & 70 & 104 \\ 48 & 111 & 148 & 207 & 120 & 182 \end{bmatrix} \approx \begin{bmatrix} 24 & 1 & 20 & 1 & 12 & 17 \\ 19 & 24 & 3 & 4 & 4 & 8 \end{bmatrix} (\text{mod } 29)$$

To do this in Excel proceed as follows

Select a blank 2×6 array and type **=MOD(** in the top leftmost cell of the array. Within the parentheses, select the array of the product matrix EP and type 29 for the divisor.





Step 5: Write out the columns of the matrix in a sequence

$$\begin{bmatrix} 24 & 1 & 20 & 1 & 12 & 17 \\ 19 & 24 & 3 & 4 & 4 & 8 \end{bmatrix}$$

These are

24 19 1 24 20 3 1 4 12 4 17 8

Replace these values by the characters from the substitution table to which these values correspond.

The encrypted message or ciphertext is **MTBYUDBEMERI**.

Decoding or deciphering the ciphertext: The steps

In this section we will try to decipher the ciphertext **MTBYUDBEMERI**

Step 1: Convert the characters to their respective Hill-2-cipher values from the substitution table

24 19 1 24 20 3 1 4 12 4 17 8

Form a 2×6 matrix using these values. Make pairs of these numbers as follows

24 19   1 24   20 3   1 4   12 4   17 8

Each pair will form a column of a 2 × 6 matrix (since there are 6 pairs). Let us call this matrix C (the ciphertext matrix)

$$C = \begin{bmatrix} 24 & 1 & 20 & 1 & 12 & 17 \\ 19 & 24 & 3 & 4 & 4 & 8 \end{bmatrix}$$

Step 2: Compute the product E^{-1} C

$$E^{-1}C = \begin{bmatrix} 12 & 16 \\ 2 & 28 \end{bmatrix}\begin{bmatrix} 24 & 1 & 20 & 1 & 12 & 17 \\ 19 & 24 & 3 & 4 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 592 & 396 & 288 & 76 & 208 & 332 \\ 580 & 674 & 124 & 114 & 136 & 258 \end{bmatrix}$$

Step 3: Reduce the product modulo 29 to obtain the substitution values.

$$\begin{bmatrix} 592 & 396 & 288 & 76 & 208 & 332 \\ 580 & 674 & 124 & 114 & 136 & 258 \end{bmatrix} \approx \begin{bmatrix} 12 & 19 & 27 & 18 & 5 & 13 \\ 0 & 7 & 8 & 27 & 20 & 26 \end{bmatrix} (\text{mod } 29)$$

The reader may perform these computations using Excel. The screenshot of the Excel sheet is as follows.

12										
13		12	16		24	1	20	1	12	17
14		2	28		19	24	3	4	4	8
15										
16					592	396	288	76	208	332
17					580	674	124	114	136	258
18										
19					12	19	27	18	5	13
20					0	7	8	27	20	26

Step 4: Write out the columns of the matrix in a sequence

$$\begin{bmatrix} 12 & 19 & 27 & 18 & 5 & 13 \\ 0 & 7 & 8 & 27 & 20 & 26 \end{bmatrix}$$

These are

12 0 19 7 27 8 18 27 5 20 13 26

Replace these values by the characters from the substitution table to which these values correspond.

The decrypted message or plaintext is **MATH_IS_FUN.**

So far we have learnt how to encrypt a plaintext using a Hill 2-cipher. This means that our encoding matrix is a 2 × 2 matrix. If we choose a 3 × 3 matrix, the plaintext will have to be converted to a 3 × n matrix (here the number of columns 'n' depends on the length of the message).

The reader is urged to try to decode the messages in the next few exercises to practice the method. All computations may be done on Excel. Note that the substitution table remains the same as before.

Exercises

(1) Decode the secret message **O? ZR ZV OW MK AC GM KX** which was encrypted using the encoding matrix

$$E = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

(2) Decode the secret message **SA_ NCN PIB WNF PRU JRP RII** which was encrypted using the encoding matrix

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

This is an example of a Hill 3-cipher.

Hint: The first step is to find the inverse of the matrix E in Z_{29}. The method described in the article for a 2 × 2 matrix may be followed. Note that the augmented matrix $[E|I]$ is the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right].$$

Conclusion

The Hill Cipher presents an interesting application of matrices and number theory to cryptography. It is open to exploration and students find it exciting to use this method. This is an example of a practical situation where performing matrix operations such as matrix multiplication and finding the inverse are actually required. It helps the student to understand the need and importance of matrix operations and also explore the method by using different keys (that is, encoding matrices). Any computing tool which can perform matrix operations, will be helpful, as the computations may be tedious and time consuming (especially when the plaintext or ciphertext is lengthy). In this article we have discussed a more general form of the method where any invertible square matrix may be chosen as the key.

References

1. http://en.wikipedia.org/wiki/Hill_cipher
2. http://practicalcryptography.com/ciphers/hill-cipher/
3. http://www.pstcc.edu/math/_files/pdf/augment.pdf (for information about the augmented matrix)

Solutions to exercises

Exercise 1:

The ciphertext **O? ZR ZV OW MK AC GM KX** converts to the hill – 2 – cipher matrix

$$\begin{bmatrix} 14 & 25 & 25 & 14 & 12 & 0 & 6 & 10 \\ 28 & 17 & 21 & 22 & 10 & 2 & 12 & 23 \end{bmatrix}$$

We pre-multiply this matrix using the inverse of the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ in Z_{29}, which is $\begin{bmatrix} 12 & 16 \\ 2 & 28 \end{bmatrix}$. Further we reduce the product modulo 29 to obtain the original values which correspond to alphabets from the substitution table. The computations are shown in MS Excel.

39												
40		12	16		14	25	25	14	12	0	6	10
41		2	28		28	17	21	22	10	2	12	23
42												
43					616	572	636	520	304	32	264	488
44					812	526	638	644	304	56	348	664
45												
46					7	21	27	27	14	3	3	24
47					0	4	0	6	14	27	0	26
48												

The values (taken column-wise) are as follows

7 0 21 4 27 0 27 6 14 14 3 27 3 0 24 26

These translate to the message

HAVE_A_GOOD_DAY.

Exercise 2:

The inverse of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ in Z_{29} is $\begin{bmatrix} 28 & 28 & 2 \\ 12 & 28 & 17 \\ 2 & 1 & 27 \end{bmatrix}$ which may be obtained as follows.

Consider the augmented matrix $[E|\,I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$.

We need to perform elementary row transformations so that E gets transformed to I, the 3×3 identity matrix, which is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We begin by performing the row operation $R_3 \to R_3 - R_1$ on $[E|\,I]$. Reducing the product modulo 29, we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & 2 & 2 & 28 & 0 & 1 \end{array}\right] (\text{mod } 29)$$

Note that the first 1 in R_3 in the original augmented matrix has been replaced by 0. In order to convert 4 to 1 (in the reduced matrix), we need to multiply it by its inverse which is 22 and perform the row operation $R_2 \to 22 \times R_2$. This gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 88 & 110 & 0 & 22 & 0 \\ 0 & 2 & 2 & 28 & 0 & 1 \end{array}\right] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 23 & 0 & 22 & 0 \\ 0 & 2 & 2 & 28 & 0 & 1 \end{array}\right] (\text{mod } 29)$$

Now, to convert the first 2 in R_3 to 0, we need to perform the row operation $R_3 \to R_3 - 2\times R_2$. We now get,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 23 & 0 & 22 & 0 \\ 0 & 0 & -44 & 28 & -44 & 1 \end{array}\right] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 23 & 0 & 22 & 0 \\ 0 & 0 & 14 & 28 & 14 & 1 \end{array}\right] (\text{mod } 29)$$

In order to convert the first 14 of R_3 to 1 (in the reduced matrix), we need to multiply it by its inverse which is 27 and perform the row operation $R_3 \to 27 \times R_3$ and reduce it modulo 29. This gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 23 & 0 & 22 & 0 \\ 0 & 0 & 378 & 756 & 378 & 27 \end{array}\right] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 23 & 0 & 22 & 0 \\ 0 & 0 & 1 & 2 & 1 & 27 \end{array}\right] (\text{mod } 29)$$

To convert 23 in R_2 to 0, we need to perform the row operation $R_2 \to R_2 - 23 \times R_2$. This gives us,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -46 & -1 & -621 \\ 0 & 0 & 1 & 2 & 1 & 27 \end{array}\right] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 12 & 28 & 17 \\ 0 & 0 & 1 & 2 & 1 & 27 \end{array}\right] (\text{mod } 29)$$

The last step is to perform the row operation $R_1 \to R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -27 \\ 0 & 1 & 0 & 12 & 28 & 17 \\ 0 & 0 & 1 & 2 & 1 & 27 \end{array}\right] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 28 & 28 & 2 \\ 0 & 1 & 0 & 12 & 28 & 17 \\ 0 & 0 & 1 & 2 & 1 & 27 \end{array}\right] (\text{mod } 29)$$

We have now succeeded in converting the augmented matrix $[E|\,I]$ to $[I|\,E^{-1}]$.

Thus, the inverse of E = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ in Z_{29} is E^{-1} =. $\begin{bmatrix} 28 & 28 & 2 \\ 12 & 28 & 17 \\ 2 & 1 & 27 \end{bmatrix}$

We can now use the matrix E^{-1} to decipher the secret message.

The ciphertext **SA_ NCN PIB WNF PRU JRP RII** converts to the hill – 3 – cipher matrix

$$\begin{bmatrix} 18 & 13 & 15 & 22 & 15 & 9 & 17 \\ 0 & 2 & 8 & 13 & 27 & 17 & 8 \\ 27 & 13 & 1 & 5 & 20 & 15 & 8 \end{bmatrix}$$

We pre-multiply this matrix using the inverse of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ in Z_{29}, which is $\begin{bmatrix} 28 & 28 & 2 \\ 12 & 28 & 17 \\ 2 & 1 & 27 \end{bmatrix}$.

Further we reduce the product modulo 29 to obtain the original values which correspond to alphabets from the substitution table. The computations are shown in MS Excel.

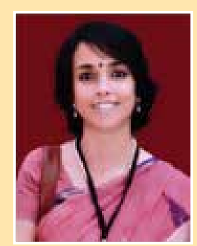
49										
50	28	28	2	18	13	15	22	15	9	17
51	12	28	17	0	2	8	13	27	17	8
52	2	1	27	27	13	1	5	20	15	8
53										
54				558	446	646	990	1216	758	716
55				675	433	421	713	1276	839	564
56				765	379	65	192	597	440	258
57										
58				7	11	8	4	27	4	20
59				8	27	15	17	0	27	13
60				11	2	7	18	17	5	26
61										

The values (taken column-wise) are as follows

7 8 11 11 27 2 8 15 7 4 17 18 27 0 17 4 27 5 20 13 26

These translate to the message

HILL_CIPHERS_ARE_FUN.



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techspace

Of Paper-folding, Geogebra and Conics – Part II

In the article Of Paper-folding, Geogebra and Conics which appeared in the Tech Space section of the July 2015 issue we had discussed the method of generating a parabola, firstly through paper-folding and then on Geogebra, a dynamic geometry software. Both methods helped us to understand and explore the various properties of a parabola. In this article we shall describe the construction of the ellipse and the hyperbola using a similar strategy of paper-folding followed by a Geogebra exploration. The reader may consider the previous article as a pre-requisite to this one.

SWATI SIRCAR

Let us recall, that the conics can be obtained by folding a family of lines which have specific properties. These lines which trace out the conic may be referred to as the *envelope* of the required conic.

By studying the way the folds are made, we can derive the equation of the conic. This repeated folding, as a point varies along a line (or a circle), is a simple low cost way of generating a locus without resorting to technology. It enables the student to get a glimpse of how a curve can be generated dynamically. We then replicate this process on GeoGebra - what is interesting is how we 'algorithmise' the paper-folding instruction in order to get the desired output on the computer.

The exploratory activities in this article include the following:

1. Paper-folding as mentioned above to get the envelopes – the process of paper-folding and the related geometry are a sheer joy!
2. Pondering over the exact points in each fold-line (or tangent to the curve) that generated it.
3. Verification using GeoGebra, plotting the actual conic and observing its formula.
4. Exploring the properties of the conic under consideration.
5. Deriving the formula for this conic.

Generating the ellipse through paper-folding

On a sheet of paper, draw a circle and mark its centre C and circumference L. Cut out the circle and mark a point P within the circle. Select a point Qon L and fold the paper so that Q coincides with P. Make a sharp crease along the fold. Now select another point Q' near Q and repeat the process. Shift the paper slightly each time to get a new point Q' on L and repeat the folding process. Note that each crease (fold line) obtained is a chord of the circle. See Figure 1.

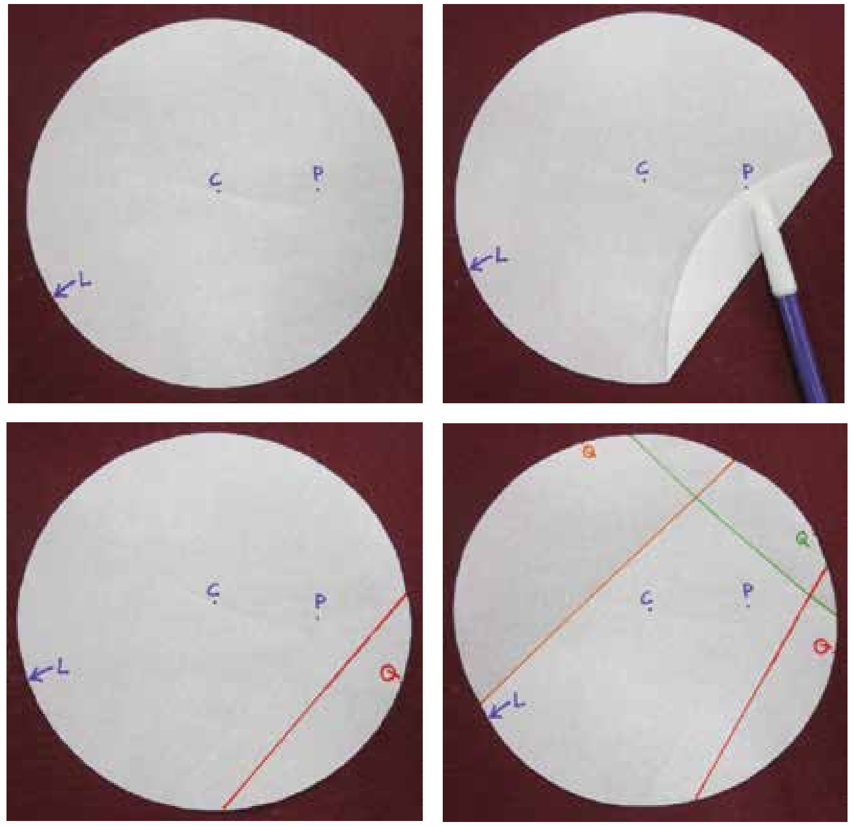


Figure 1.

The emerging shape, which is an ellipse, is clearly visible in Figure 2.

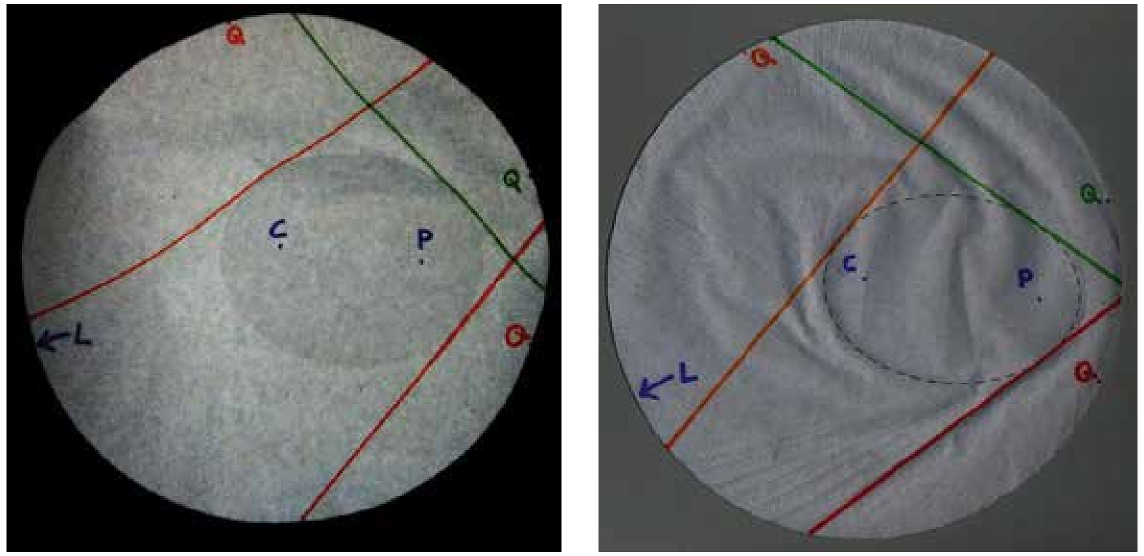


Figure 2.

Observe that the first fold line (shown in green in Figure 2) is the perpendicular bisector of PQ. As the position of Q changes on L, new fold lines are generated and these are tangents to the curve. Let us pick any point Q on L and draw the corresponding fold line. Let us mark the point of intersection of CQ with the fold line as Q_1.

The curve is the locus of Q_1 as Q varies along L!

Why is this?

The ellipse is defined as *the locus of all points the sum of whose distances from two given points is a constant.* These two given points are called the foci of the ellipse.

Now, $Q_1P = Q_1Q$ by symmetry, therefore $CQ_1 + Q_1P = CQ_1 + Q_1Q = CQ$, i.e. radius of the given circle, which is a constant, independent of Q. So the elliptical curve is the locus of Q_1 as Q varies along L with C and P as its foci!

The ellipse has two axes or lines of symmetry. These are the major axis and minor axis (indicated as A_1A_2 and B_1B_2 respectively in Figure 3). These may be obtained as follows:

Join CP and extend it to cut the circle at A_{1Q} and A_{2Q}. This line meets the ellipse in A_1 and A_2, (as shown in figure 3 and these are referred to as the vertices or end points of the ellipse. Note that $A_1$and $A_2$are respectively the mid-points of PA_{1Q} and PA_{2Q}. To find the endpoints B_1 and B_2 of the ellipse, we use the fact that at B_1 and B_2, the tangents are parallel to CP. Through P, draw a line perpendicular to CP and cutting the circle at points B_{1Q} and B_{2Q}. In the triangle $B_{1Q}CP$, the perpendicular bisector of $B_{1Q}P$ (the fold line or tangent) is parallel to the base CP and hence passes through the mid-point of CB_{1Q}. Hence the vertices at the ends of the minor axes are the mid-points B_1, B_2 of CB_{1Q} and CB_{2Q}. Having got the four vertices of the ellipse (see Figure 3), we can now map the arcs A_1B_1, B_1A_2, A_2B_2 and B_2A_1 on the ellipse which correspond to points on the four arcs of L. Note how $B_{1Q}A_{2Q}$ is much shorter than $A_{1Q}B_{1Q}$ even though B_1A_2 and A_1B_1 are of the same length.

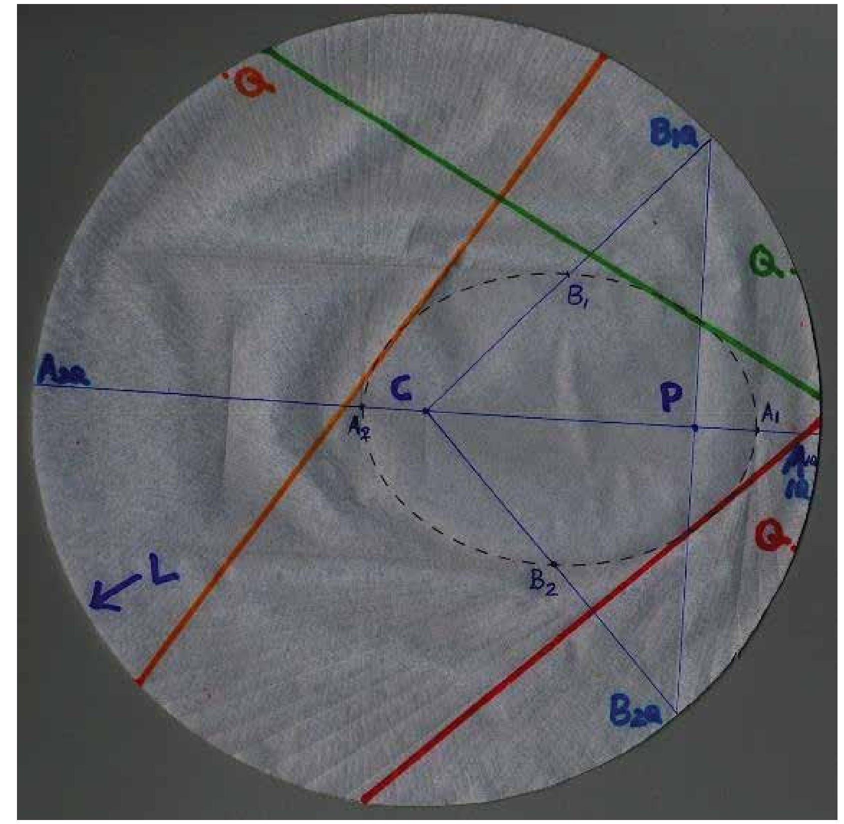


Figure 3.

Let us now derive the equation of the ellipse. Take the origin to be the centre of the ellipse i.e., the midpoint of CP. Then $C = (-a, 0)$ and $P = (a, 0)$ for some $a > 0$. Let r be the radius of the circle; note that $r > 2a$.

The equation of the circle is $(x + a)^2 + y^2 = r^2$.

Let the coordinates of Q be (u, v). This gives

$$(u + a)^2 + v^2 = r^2 \tag{1}$$

Note that $A_1 = (r/2, 0)$ as it is the mid-point of $A_{1Q}(r - a, 0)$ and $P(a, 0)$. Similarly, $A_2 = (-r/2, 0)$,

Since OB_1 is ½ PB_{1Q}, and $CP = 2a$, $CB_{1Q} = r$, $OB_1 = 1/2\ \sqrt{(r^2 - 4a^2)})$, so B_1 is the point $(0, 1/2\ \sqrt{(r^2 - 4a^2)})$. Similarly B_2 is the point $(0, -1/2\ \sqrt{(r^2 - 4a^2)})$.

Let $\alpha = r/2$ and $\beta = 1/2\ \sqrt{(r^2 - 4a^2)} \Rightarrow r^2 = 4\alpha^2$ and $4\beta^2 = r^2 - 4a^2$

$\therefore A_1 = (\alpha, 0),\ A_2 = (-\alpha, 0),\ B_1 = (0, \beta)$ and $B_2 = (0, -\beta)$

We next calculate the equation of the fold line using (i) the midpoint of PQ, (ii) slope of PQ and (iii) relation between the slopes of two perpendicular lines.

$$\Rightarrow y - \frac{v}{2} = \frac{a - u}{v} \times \left(x - \frac{u + a}{2}\right) \tag{2}$$

Also, the equation of CQ is:

$$y = \frac{v}{u + a} \times (x + a) \tag{3}$$

We now use (1), (2) and (3) to get

$$\frac{y}{v} = \frac{x + a}{u + a} = \frac{r^2 - 4a^2}{2(v^2 + u^2 - a^2)} = \frac{2\beta^2}{v^2 + u^2 - a^2} \tag{4}$$

$$y = \frac{2\beta^2 v}{v^2 + u^2 - a^2} \Rightarrow \frac{y}{\beta} = \frac{2\beta v}{v^2 + u^2 - a^2} \quad (5)$$

and

$$x + a = \frac{(u+a)\left(r^2 - 4a^2\right)}{2(v^2 + u^2 - a^2)} \Rightarrow x = \frac{2\alpha^2 u - 2\alpha^2 a}{v^2 + u^2 - a^2} = \frac{2\alpha^2 (u-a)}{v^2 + u^2 - a^2} \Rightarrow \frac{x}{\alpha} = \frac{2\alpha\,(u-a)}{v^2 + u^2 - a^2} \quad (6)$$

In order to eliminate x and y, from (5) and (6)

$$\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 = \frac{4\alpha^2\,(u-a)^2 + 4\beta^2 v^2}{\left(u^2 + v^2 - a^2\right)^2}$$

Now

$$\begin{aligned} 4\alpha^2(u-a)^2 + 4\beta^2 v^2 &= r^2(u-a)^2 + (r^2 - 4a^2)v^2 \\ &= (\mathrm{A}+\mathrm{B})^2 - 4\mathrm{AB} \text{ where } \mathrm{A} = u^2 + v^2 \text{ and } \mathrm{B} = a^2 \\ &= (\mathrm{A}-\mathrm{B})^2 = (u^2 + v^2 - a^2)^2 \\ &\Rightarrow \left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 = 1 \end{aligned}$$

The complete derivation is available on http://teachersofindia.org/en/periodicals/at-right-angles

Working with GeoGebra

Dynamic Geometry Software such as GeoGebra proves invaluable for mathematical investigations. (It is available from http://www.geogebra.org/download.) While most students are very comfortable with hands-on activities, the tedium of repeated and careful folding is eliminated with the use of technology. Patterns emerge faster and can easily be viewed with the help of the 'Trace' button and the judicious use of colour. This enables the student to focus on the mathematics of the investigation rather than the technicalities of the activity. The activity being studied here assumes that the reader has familiarity with the main features of Geogebra. To convert the activity into a Geogebra exploration and to select the appropriate commands, the student needs to ask the following questions:

1. What is the outcome?
2. What is the mathematical aspect to this physical activity?
3. How can I give this command?

For example, in order to replicate the steps "*Next, select a point on L, fold that point to P, and crease the paper along the fold…………*" the student should arrive at the following answers:

1. What is the outcome? **The point on *L* should coincide with P**.
2. What is the mathematical aspect to this physical activity? **P should be the reflection of the point on *L*.**
3. How can I give this command? **The crease on the paper is the mirror for the reflection of the point on *L* so that it coincides with P. So the crease is the perpendicular bisector of the line joining the point on *L* to P.**

In GeoGebra, the point Q can be easily moved on L with the use of the arrow key. Further, the use of sliders allows the student to observe changes in the ellipse as the radius of the circle and the distance between the two points change.

1. Define 2 sliders r (varying between 0 and 12) and a (varying between 0 and r/2), note that this automatically ensures $r \geq 2a$.
2. Plot $\mathrm{C} = (-a, 0)$ and $\mathrm{P} = (a, 0)$, $\mathrm{A_{1Q}} = (r - a, 0)$ on the positive x-axis [type A_{1Q} for $\mathrm{A_{1Q}}$].
3. Construct circle L centred at C and passing through $\mathrm{A_{1Q}}$.
4. Take any point Q on L.
5. Construct the perpendicular bisector of PQ.

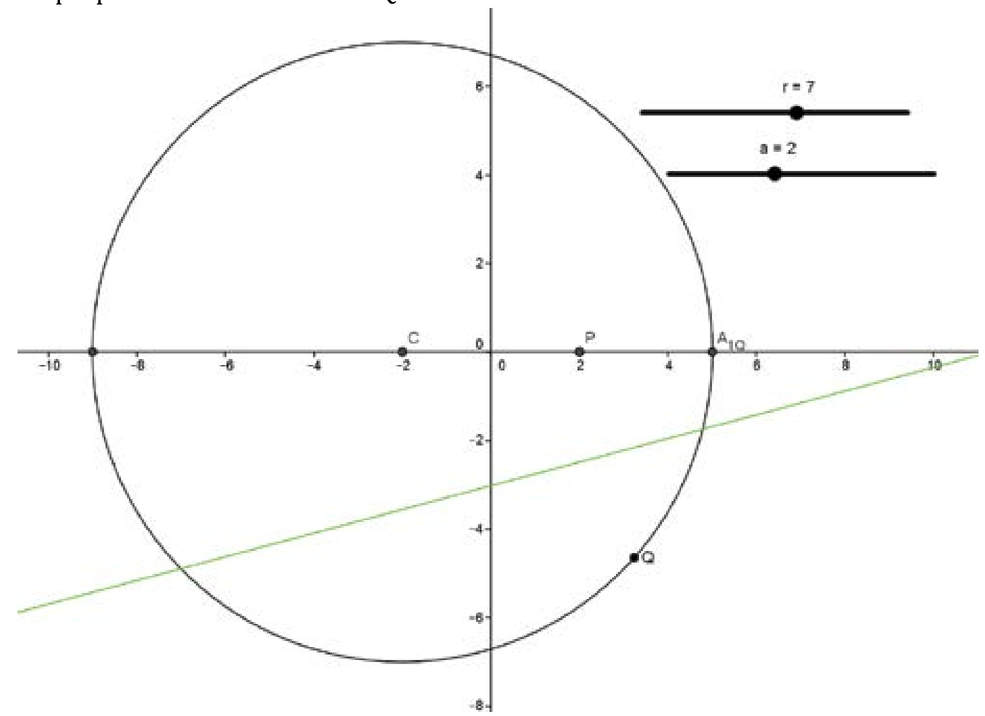


Figure 4.

6. Move the point Q on L. Observe that the perpendicular bisector traces an ellipse. The ellipse may be obtained by activating the trace option of the perpendicular bisector and moving the point Q along L.

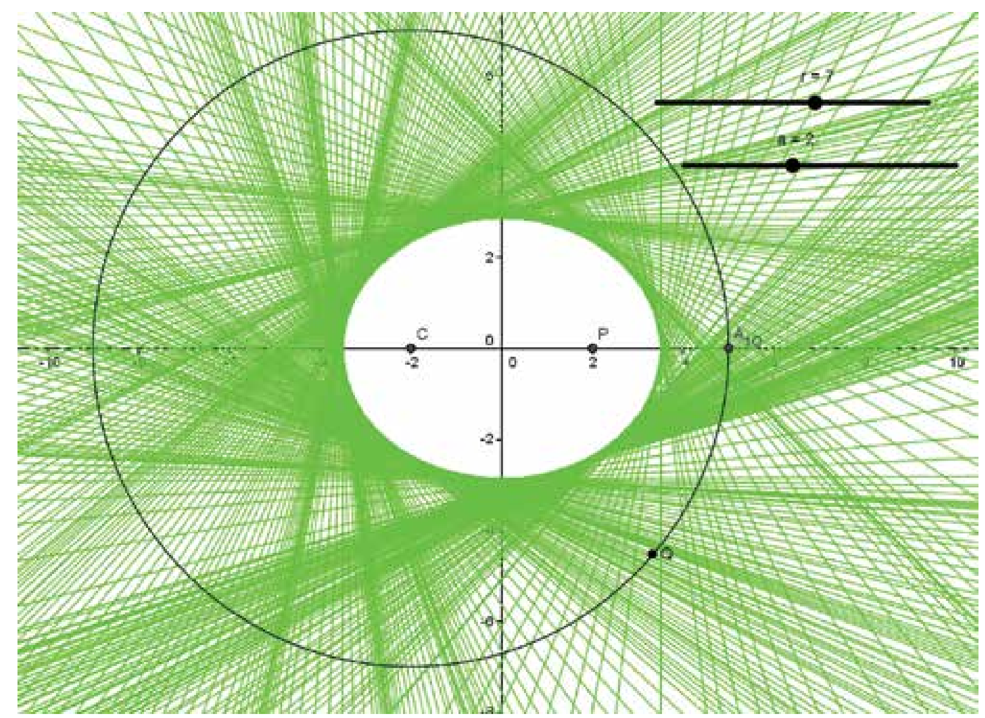


Figure 5.

7. Undo trace

8. Draw the line segment CQ
9. Get the intersection between CQ and the perpendicular bisector of PQ i.e. Q_1

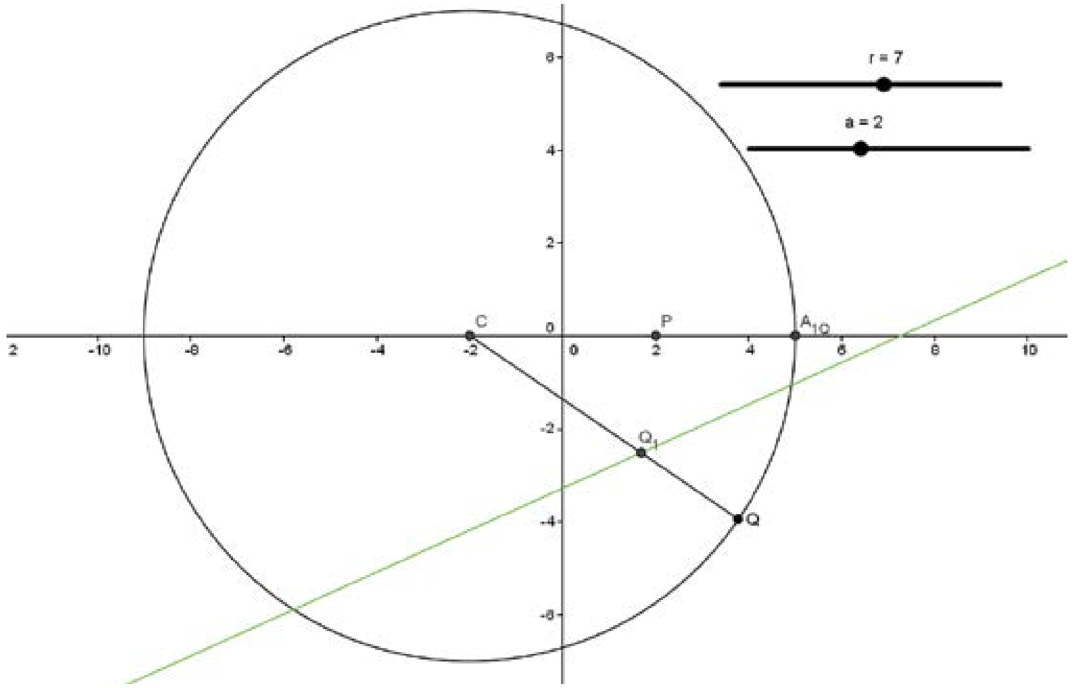


Figure 6.

10. Use trace on both Q_1 and the perpendicular bisector to verify that Q_1 indeed is the point on the tangent

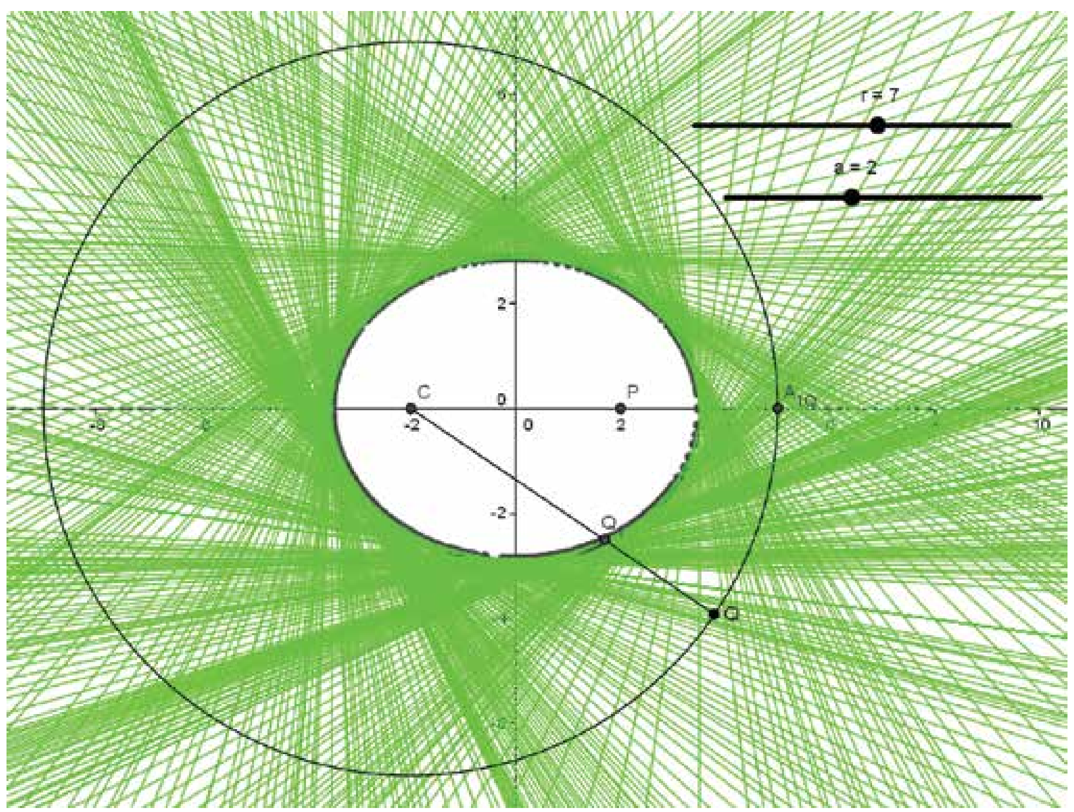


Figure 7.

11. Undo both traces
12. Construct the midpoint A_1 of P and A_{1Q}

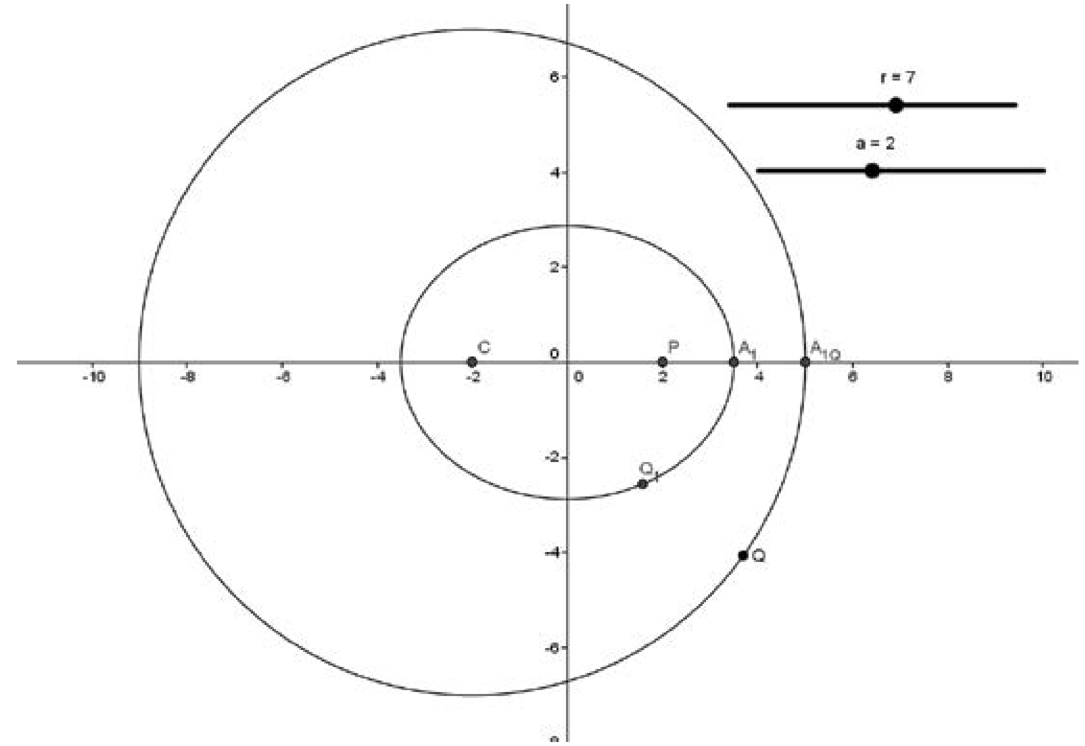


Figure 8.

13. Mark the point of intersection A_{2Q} of L and the negative x-axis.
14. Construct the midpoint A_2 of P and A_{2Q}.
15. Construct the line perpendicular to x-axis through P.
16. Mark both intersection points B_{1Q} and B_{2Q} of the above line with circle L.
17. Mark midpoints B_1 and B_2 of CB_{1Q} and CB_{2Q} respectively.
18. Mark midpoint O of CP.
19. Generate the ellipse by selecting the draw ellipse option from the tool bar; the points C, P and A_1 may be used in this case.
20. Verify that Q_1 actually moves along this ellipse.
21. Vary the sliders (of course ensuring that $r > 2a$) to observe the corresponding change in the ellipse and its equation.

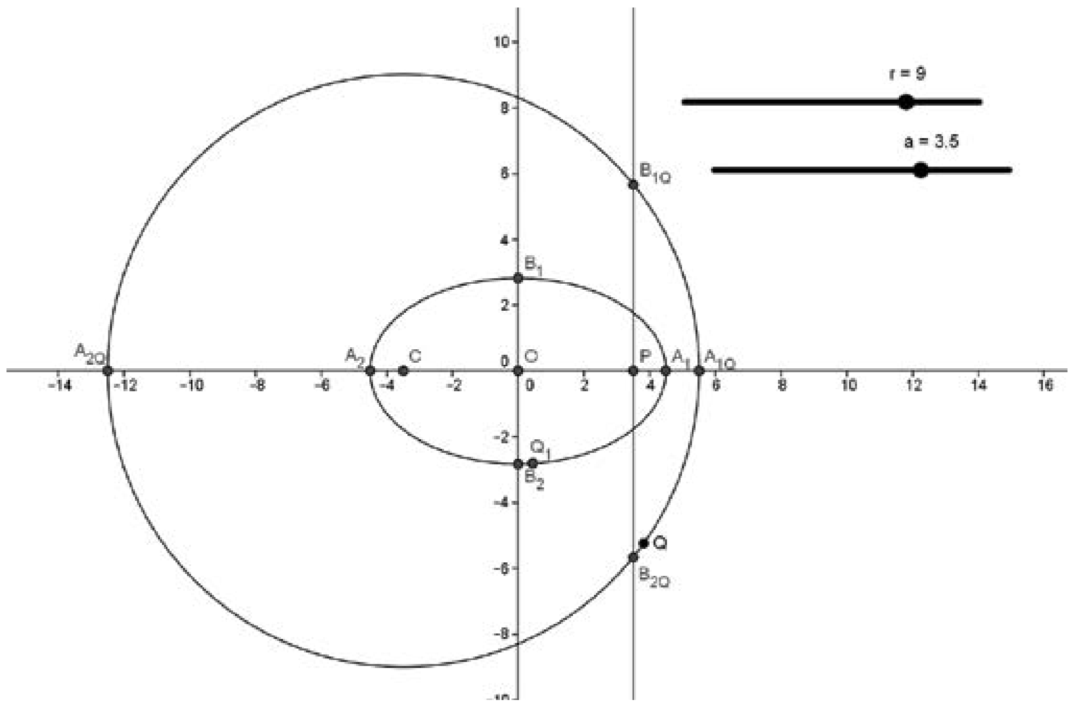


Figure 9.

In the above Geogebra construction we obtain a family of ellipses with major axis lying on the x-axis. An ellipse with centre at the origin and major axis on x-axis is usually referred to as the 'standard form' of the ellipse. An interesting task for students would be to generate ellipses with the major axis along the y-axis and centres which do not lie at the origin. The equations corresponding to these non-standard forms may also be studied – it is also instructive to key in an equation for the ellipse and predict its orientation, centre and so on. Thus GeoGebra can work as a self-assessment tool and is useful for the student to gauge one's understanding of the topic. Not just this, by observing the student's predicted outcome and the actual outcome, the teacher will be able to quickly identify the student's difficulties and address specific instead of general problems. The use of dynamic geometry software thus has clear pedagogical benefits.

Generating the Hyperbola

This is very similar to the ellipse, including the calculations involved. The difference is that the point P is taken outside the circle, we do not cut out the circle, and L is no longer an edge of the paper. The paper needs to be semi-transparent for P to be visible through an extra layer of the paper as one tries to fold L to P. Hence butter-paper is recommended.

However, if you do want to brave it out with regular paper, then poke at P to get an imprint on the other side of the paper. Draw P and a thick, small circle around it with a pen. Now fold so that L passes through P. Thanks to the small thick circle, P should be easier to see. If you are still having difficulty, hold up the folded paper against light to locate P. You may have to hold the paper up for each fold.

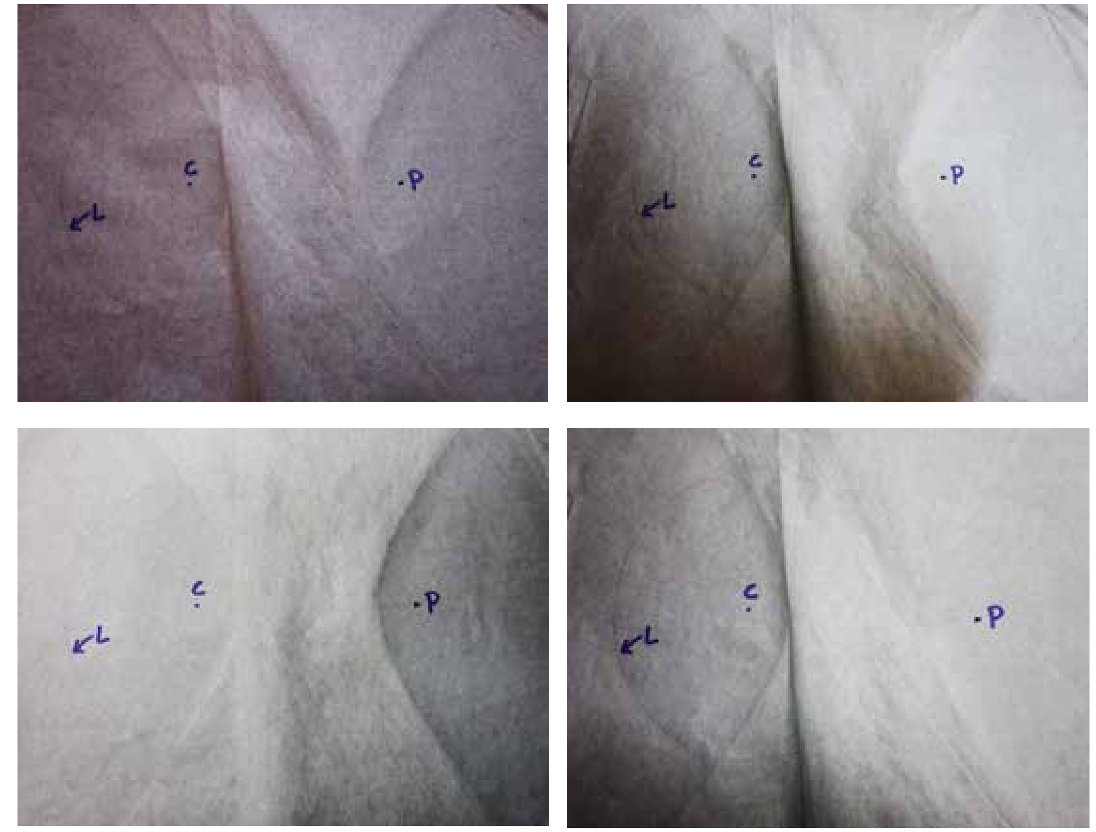


Figure 10.

The Geogebra steps are also the same except for the initial sliders:

1. Define 2 sliders a (varying between 0 and 6) and r (varying between 0 and $2a$), note that this ensures $r \leq 2a$.

   The remaining steps remain the same.

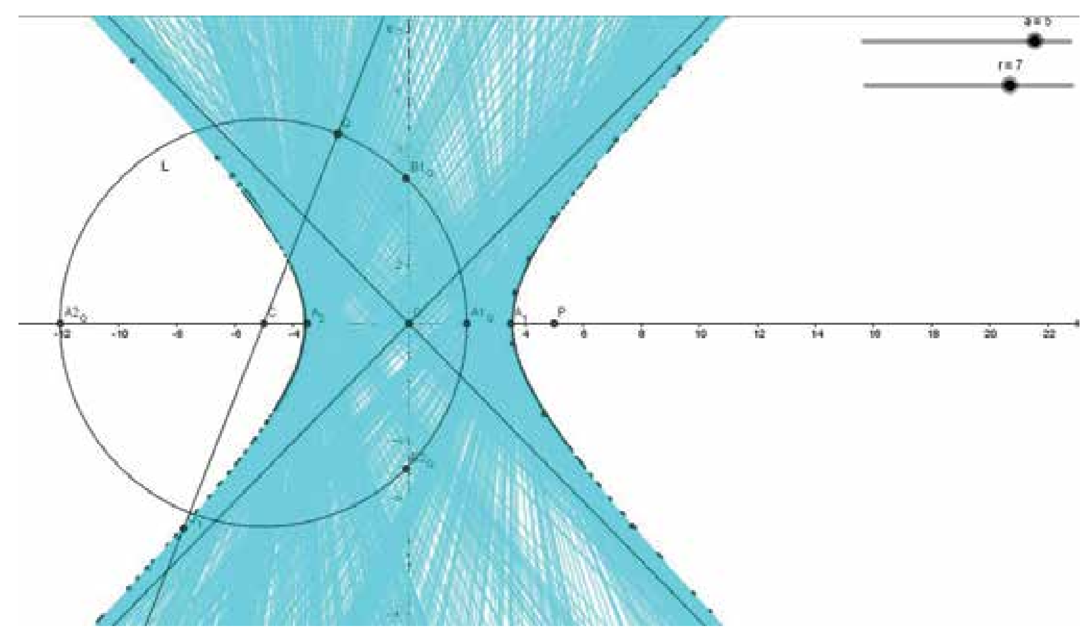


Figure 11.

It is worth noting the positions of Q for which the perpendicular bisector of PQ and the radial line (not a line segment) CQ become parallel. These perpendicular bisectors (or the corresponding fold lines) are the asymptotes of the hyperbola.

Conclusion

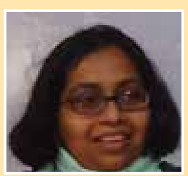
Ellipses are most commonly encountered in the orbits of celestial bodies, e.g. the Earth around the Sun or the Moon going around the Earth. All artificial satellites also move in elliptical orbits with the Earth at one focus. However, they are encountered even earlier, say when we try to draw any circular vessel. A circle when viewed at an angle appears as an ellipse. So the top edge of a circular vessel is usually drawn as this conic. Like the parabola, the ellipse also has reflective properties which are made use of by architects to construct whispering galleries. Any wave transmitted from one focus will travel through the second focus after reflection off an elliptical wall. An ellipse occurs as the intersection when a cylinder and a plane cross each other at an angle. This is useful in fitting pipes vertically on a sloping roof.

The hyperbola on the other hand can be seen in the shadow cast by a torch or a cylindrical lamp shade. Cooling towers of nuclear plants have hyperbolic vertical cross sections. When stones are thrown in a pond, the two sets of circular waves intersect along a hyperbola.

Whereas circles and straight lines can easily be drawn, it is not as easy to draw ellipses or hyperbolas on a sheet of paper. The paper-folding activity generates these curves and the underlying geometry is instrumental in understanding the geometry of the shapes formed. GeoGebra provides an additional layer of understanding. Also, GeoGebra can help one predict the formula and the parameters involved. And all that can be linked to the folds and proved using algebra!

References

1. *Mathematics Through Paper-folding*, Alton T. Olson, University of Alberta
2. http://www3.ul.ie/~rynnet/swconics/practical_applications1.htm
3. http://www.pleacher.com/mp/mlessons/calculus/



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problem corner

Two Problems

C⊗MαC

We present this time a rather small collection of problems (just two), followed by their solutions. We state the problems first so you have a chance to try them out on your own.

Problems

(1) Is it possible to arrange the numbers 1, 2, 3, ..., 15, 16 in a sequence such that the following property is satisfied: *Each pair of consecutive numbers adds up to a perfect square*?

(2) Let P be a variable point inside a given triangle ABC, and let D, E, F be the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Find all P for which $BC/PD + CA/PE + AB/PF$ is least.
[Adapted from Problem 1 of the 22nd IMO, held in the USA in 1981]

Solutions

Problem 1. *Is it possible to arrange the numbers 1, 2, 3, ..., 15, 16 in a sequence so that each pair of consecutive numbers adds up to a perfect square?*

We shall show that this is possible by actually constructing such a sequence.

Let us assume that it is possible to do this, and see where this hypothesis takes us. The least possible sum of two numbers from the set is 3, and the largest possible sum is 31. So each pair of consecutive numbers in the sequence must add up to one of the following numbers: 4, 9, 16, 25. Using these facts, we list the possible neighbours of each number in the set, as shown below:

Number	Possible companions
1	3, 8, 15
2	7, 14
3	1, 6, 13
4	5, 12
5	4, 11
6	3, 10
7	2, 9
8	1
9	7, 16
10	6, 15
11	5, 14
12	4, 13
13	3, 12
14	2, 11
15	1, 10
16	9

We notice that the numbers 8 and 16 have just one possible neighbour each. This tells us right away that if such a sequence is at all possible, then 8 and 16 must lie at its two ends. We now start building the sequence by placing 8 and 16 at the two ends and working our way inwards; say 8 at the left-hand end, and 16 at the right-hand end. The table displayed above shows that 8 must be followed by 1, and 16 must be preceded by 9. Since the possible neighbours of 9 are 7 and 16, the number preceding 9 must be 7. This must be preceded in turn by 2, then 14, then 11, then 5, then 4, then 12, then 13, then 3, then 6 (since 1 is not available, having been already 'used up'), then 10, then 15, and now the whole sequence is complete. Here is the final sequence:

8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7, 9, 16.

If we add up consecutive pairs of numbers of this sequence, here are the sums that we get:

9, 16, 25, 16, 9, 16, 25, 16, 9, 16, 25, 16, 9, 16, 25.

The sequence has a curious symmetry about it. Readers may want to explore the patterns further.

The problem-solving strategy used to solve this problem should be noted. We did not use any "advanced" techniques; we only followed the consequences of the stated property and listed the various possibilities, and this led us to the answer.

Problem 2. *Let P be a variable point inside a given triangle ABC, and let D, E, F be the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Find all P for which $BC/PD + CA/PE + AB/PF$ is least.*

Let a, b, c denote the sides BC, CA, AB of the triangle, and let $PD = u$, $PE = v$, $PF = w$. We must minimise the quantity $a/u + b/v + c/w$. There are three variables occurring in this problem: u, v, w. This may make the problem appear daunting. However, the three variables are not independent of each other. By drawing the segments connecting P to the vertices of the triangle, it is easy to see that the areas of $\triangle PBC$, $\triangle PCA$ and $\triangle PAB$ are $au/2$, $bv/2$ and $cw/2$, respectively; see Figure 1 (i). The sum of these three areas must be equal to the area Δ of $\triangle ABC$, which is a constant. Hence we have:

$$au + bv + cw = 2\Delta.$$

So u, v, w are connected by a linear relation. We must minimise $a/u + b/v + c/w$ subject to this relation. Note that all quantities occurring in the problem are positive, as P is assumed to lie in the interior to the triangle.

We simplify the problem by fixing one of the variables, say u, and allowing only the other two variables (v and w) to vary. If u has a constant value, then P is being constrained to move along a line parallel to side BC; see Figure 1 (ii). We now examine how to minimise $b/v + c/w$ as P moves on this line segment. Algebraically, the problem is the following: minimise $b/v + c/w$, subject to $bv + cw = k$, where k is a constant. Note that this is a two-variable problem. We solve it graphically using the following artifice. Let $y = b/v$ and $z = c/w$. Then the problem is, in terms of the new variables:

Minimise $y + z$,

subject to the condition: $\dfrac{b^2}{y} + \dfrac{c^2}{z} = k.$

Now we examine the situation on a (y, z) graph. The statement that $b^2/y + c^2/z = k$ means that the points (y, z) under consideration all lie on a certain hyperbola $\mathcal{H}$; see Figure 2. To minimise $y + z$ subject to the point (y, z) lying on this hyperbola means that we must imagine the line $y + z =$ constant being moved parallel to itself till it is exactly tangent to the hyperbola. Then the

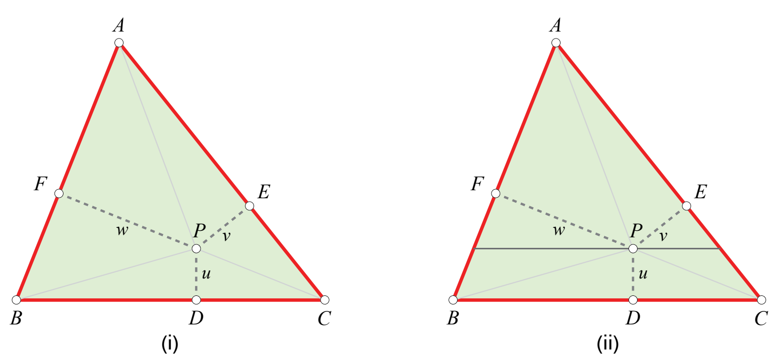


Figure 1.

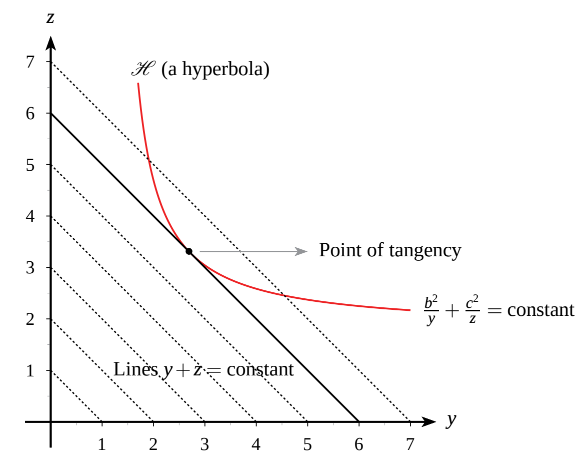


Figure 2.

coordinates of the point of tangency give us the desired values of y and z.

The slope of the tangent line is, of course, -1. Hence we must find a point of the curve at which the slope is -1. For this, we use differentiation. We have:

$$\frac{b^2}{y} + \frac{c^2}{z} = 1, \quad \therefore \frac{dz}{dy} = -\frac{b^2 z^2}{c^2 y^2}, \quad \therefore \frac{b^2 z^2}{c^2 y^2} = 1,$$

since the slope is -1. Hence $bz = cy$, giving $b/y = c/z$ and therefore $v = w$. Therefore, the optimising point is the one at which the distances from the sides AB and AC are equal. This means that the point lies on the internal bisector of angle BAC. And this conclusion holds independent of the value we assign to u. That is, for each value of u, the optimising point lies on the internal bisector of angle BAC.

By symmetry, it follows that the optimising point lies on each of the three internal angle bisectors of the triangle. Hence the optimising point is the incentre of the triangle.

We now offer a second solution of this problem. This is an extremely different approach, but it is well worth studying closely. We shall make use of a very famous result known as the **Cauchy–Schwartz inequality**. As you may not be familiar with it, we shall make a few remarks about it before applying it to the problem at hand. For more on the topic, please see [1] and [2].

In the two-dimensional coordinate plane, let $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j}$ be two vectors; here, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x- and

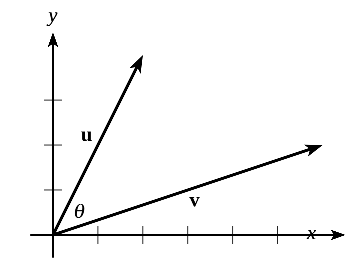


Figure 3.

y-directions respectively; see Figure 3. Their scalar product ("dot product") is then equal to

$$\mathbf{u} \cdot \mathbf{v} = (a_1\mathbf{i} + a_2\mathbf{j}) \cdot (b_1\mathbf{i} + b_2\mathbf{j}) = a_1 b_1 + a_2 b_2.$$

By the definition of scalar product, we also have:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta,$$

where θ is the angle between the two vectors. Here $|\mathbf{u}|$ and $|\mathbf{v}|$ are the lengths of the two vectors, given by:

$$|\mathbf{u}| = \sqrt{a_1^2 + a_2^2},$$

$$|\mathbf{v}| = \sqrt{b_1^2 + b_2^2}.$$

Hence we have:

$$\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}\,\cos\theta = a_1 b_1 + a_2 b_2,$$

and so:

$$\cos^2\theta = \frac{(a_1 b_1 + a_2 b_2)^2}{(a_1^2 + a_2^2)\,(b_1^2 + b_2^2)}.$$

Now $\cos^2\theta \le 1$, with equality if and only if $\theta = 0$ or π. This implies that

$$\boxed{(a_1 b_1 + a_2 b_2)^2 \le (a_1^2 + a_2^2)\,(b_1^2 + b_2^2),}$$

with equality if and only if the two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel to each other, which is the case precisely when (a_1, a_2) is a multiple of (b_1, b_2). This result is known as the 'Cauchy–Schwartz inequality'.

This reasoning works equally well in three dimensions. Thus, if $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are two vectors in three dimensions, then by considering their dot product, we arrive at the following result:

$$\boxed{\begin{aligned}(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \le {} & (a_1^2 + a_2^2 + a_3^2) \\ & (b_1^2 + b_2^2 + b_3^2),\end{aligned}}$$

with equality if and only if the two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel to each other, which is the case precisely when (a_1, a_2, a_3) is a multiple of (b_1, b_2, b_3).

The Cauchy–Schwartz inequality is widely used in higher mathematics, and is particularly valued by 'mathletes' preparing for the Mathematical Olympiads. Let us now show how it can be applied to the problem at hand.

We must minimise the quantity $a/u + b/v + c/w$, with notation as defined earlier. Drawing on the fact that $au + bv + cw$ is a constant for all points P (it is equal to twice the area of $\triangle ABC$), we could equivalently express the problem as: minimise the quantity

$$(au + bv + cw)\left(\frac{a}{u} + \frac{b}{v} + \frac{c}{w}\right).$$

The form of the above product acts as an instant trigger, telling us to invoke the Cauchy–Schwartz inequality. So that is what we will do now, but before we do so, we must "cook" the problem a bit. We define the following two vectors $\mathbf{p}$ and $\mathbf{q}$:

$$\mathbf{p} = \sqrt{au}\,\mathbf{i} + \sqrt{bv}\,\mathbf{j} + \sqrt{cw}\,\mathbf{k},$$

$$\mathbf{q} = \sqrt{\frac{a}{u}}\,\mathbf{i} + \sqrt{\frac{b}{v}}\,\mathbf{j} + \sqrt{\frac{c}{w}}\,\mathbf{k}.$$

By the Cauchy–Schwartz inequality we have:

$$\mathbf{p} \cdot \mathbf{q} \le |\mathbf{p}|\,|\mathbf{q}|,$$

with equality if and only if $\mathbf{p}$ is parallel to $\mathbf{q}$.

Now we have, for these two vectors:

$$\mathbf{p} \cdot \mathbf{q} = a + b + c,$$

$$|\mathbf{p}| = \sqrt{au + bv + cw} = \sqrt{2\Delta},$$

$$|\mathbf{q}| = \sqrt{\frac{a}{u} + \frac{b}{v} + \frac{c}{w}}.$$

Hence we have:

$$\sqrt{\frac{a}{u} + \frac{b}{v} + \frac{c}{w}} \times \sqrt{2\Delta} \ \ge\ a + b + c,$$

with equality if and only if

$$\frac{\sqrt{au}}{\sqrt{a/u}} = \frac{\sqrt{bv}}{\sqrt{b/v}} = \frac{\sqrt{cw}}{\sqrt{c/w}},$$

i.e., if and only if $u = v = w$. This condition defines the incentre of the triangle (as it indicates that the point is equidistant from the three sides of the triangle). Hence we have:

$$\frac{a}{u} + \frac{b}{v} + \frac{c}{w} \ge \frac{(a+b+c)^2}{2\Delta},$$

with equality if and only if P coincides with the incentre of the triangle. So the optimising point is the incentre of the triangle, and the least value assumed by $a/u + b/v + c/w$ is

$$\frac{(a+b+c)^2}{2\Delta} = \frac{4s^2}{2\Delta} = \frac{2s}{r},$$

where s and r are respectively the semi-perimeter and in-radius of the triangle.

Observe how the use of this inequality leads to a quick and straightforward solution of the problem. The trick of course lies in defining the two vectors appropriately.

Remark: A link between the Cauchy–Schwartz inequality and the Pearson correlation coefficient. Before closing, we draw attention to a link between the above and an apparently unconnected topic. Those of you who have studied mathematics at the higher secondary level will know of the Pearson correlation coefficient defined for bivariate data sets. To jog your memory: let (x_i, y_i) for $i = 1, 2, \ldots, n$ be n bivariate date pairs for some population. Here X and Y are two attributes associated with the individuals of the population. We now define the following three quantities: the covariance of X and Y,

$$\operatorname{Cov}(X, Y) = \frac{\sum_i (x_i - \bar{x})\,(y_i - \bar{y})}{n},$$

and the two individual variances:

$$\operatorname{Var}(X) = \frac{\sum_i (x_i - \bar{x})^2}{n}, \qquad \operatorname{Var}(Y) = \frac{\sum_i (y_i - \bar{y})^2}{n},$$

all summations being over $i = 1, 2, \ldots, n$. Then the Pearson correlation coefficient r is defined to be:

$$r = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X)}\,\sqrt{\operatorname{Var}(Y)}}.$$

At this point, most textbooks at the school level assert without proof: "It can be proved that $-1 \le r \le 1$." We shall show here that the inequalities for r follow from the Cauchy–Schwartz inequality. After squaring, the inequality for r assumes the following form:

$$\big(\operatorname{Cov}(X, Y)\big)^2 \le \operatorname{Var}(X)\,\operatorname{Var}(Y).$$

Let $a_i = x_i - \bar{x}$ and $b_i = y_i - \bar{y}$. Then we need to prove the following:

$$\left(\frac{\sum_i a_i b_i}{n}\right)^2 \le \frac{\sum_i a_i^2}{n}\,\frac{\sum_i b_i^2}{n}.$$

Multiplying through by n^2, this assumes the following form:

$$\left(\sum_i a_i b_i\right)^2 \le \left(\sum_i a_i^2\right) \cdot \left(\sum_i b_i^2\right).$$

What we have is exactly the statement of the Cauchy–Schwartz inequality. Hence the statement that $r^2 \le 1$ follows from the Cauchy–Schwartz inequality. As we have already proved this inequality, the claim about the correlation coefficient follows.

References

1. https://en.wikipedia.org/wiki/Cauchy-Schwarz_inequality
2. http://www.artofproblemsolving.com/wiki/index.php/Cauchy-Schwarz_Inequality

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

problem corner

Problems for the Middle School

Problem Editor : R. ATHMARAMAN

Problems for Solution

Problem IV-3-M.1
A number when increased by its cube results in the number 592788. Find the number.

Problem IV-3-M.2
Find the two prime factors of 206981 given that one of them is approximately three times the other.

Problem IV-3-M.3
How would you distribute 44 pencils to 10 students such that each student receives a different number of pencils?

Problem IV-3-M.4
Find the sum of all three-digit numbers $\overline{ABC}$ such that the two-digit numbers $\overline{AB}$ and $\overline{BC}$ are both perfect squares.

[Jamaican Math Olympiad 2015]

Problem IV-3-M.5
The numbers $1, 2, 3, 5, 7, 11, 13$ are written on a board. You may erase any two numbers a and b and replace them by the single number $ab + a + b$. After repeating this process several times, only one number remains on the board. What might be this number?

[Adapted from UAB MTS: 2006-2007]

Problem IV-3-M.6
Between 3 PM and 4 PM, Ramya looked at her watch and noticed that the minute hand was between 5 and 6. When she looked next at the watch, slightly less than two hours later, she noticed that the hour and minute hands had switched places. What time was it when she looked at the watch the second time?

[Adapted from "Mathematical Wrinkles" by S.I. Jones, 1912]

Solutions of Problems in Issue-IV-2 (July 2015)

Solution to problem IV-2-M.1
(a) Find the sum of the prime divisors of 2015.

Since $2015 = 5 \times 13 \times 31$, the required sum is $5 + 13 + 31 = 49$.

(b) Find another number for which the sum of the prime divisors is the same.

We only need a collection of primes whose sum is 49. There are many such collections, for example: $7, 19, 23$, giving the number $7 \times 19 \times 23 = 3059$.

Another such collection is $3, 17, 29$, giving the number $3 \times 17 \times 29 = 1479$.

Comment. The statement of the problem allows for some degree of ambiguity if we permit repeated primes in the prime factorisation of the number. For example, what is the sum of the prime divisors of 18? Is it $2 + 3 = 5$, or is it $2 + 3 + 3 = 8$? Obviously, there is no 'is' about the answer; it depends on which interpretation we decide to follow.

If the latter interpretation is followed, then the problem has solutions like $13 + 13 + 23$, giving the number $13 \times 13 \times 23 = 3887$.

Solution to problem IV-2-M.2
The sum of the digits of a natural number n is 2015. Can n be a perfect square?

No. To prove this, we use the test for divisibility by 3 ("The remainder that a number leaves on division by 3 is equal to the remainder that its sum of digits leaves on division by 3"), and the fact that on division by 3, every square number leaves remainder 0 or 1 (i.e., no square is of the form $3k + 2$). Now observe that 2015 is of the form $3k + 2$. (Its sum of digits is 8 which is of the form $3k + 2$. Or, more directly: $2015 = 3 \times 671 + 2$.) Hence n too is of the form $3k + 2$. Invoking the fact noted above, we deduce that n is not a perfect square.

Solution to problem IV-2-M.3
Is there a five-digit perfect square such that when 1 is added to each digit, the answer is again a perfect square? Assume that the addition of 1 to each digit starts from the units end and proceeds leftwards, with carry.

Let a^2 be the five-digit perfect square; then we have $a^2 + 11111 = b^2$ (i.e., another perfect square; of course, $b > a$). This yields: $b^2 - a^2 = 11111 = 41 \times 271$. (Yes, we do need to work out this factorisation!) This yields: $(b - a) \times (b + a) = 1 \times 11111 = 47 \times 271$, hence either $b - a = 1$ and $b + a = 11111$, or $b - a = 41$ and $b + a = 271$. The former possibility yields:

$$b = \frac{1}{2}(11111 + 1) = 5556,$$
$$a = \frac{1}{2}(11111 - 1) = 5555,$$

giving $a^2 = 5555^2 = 30858025$. But this is certainly not a five-digit number. So it cannot be the answer we seek.

The second possibility yields:

$$b = \frac{1}{2}(271 + 41) = 156,$$
$$a = \frac{1}{2}(271 - 41) = 115,$$

giving $a^2 = 115^2 = 13225$. Observe that if we add 1 to each digit of this number, starting from the units end, we get 24336, which equals 156^2. Hence the sought-after answer is: 13225.

Solution to problem IV-2-M.4
The sum of three integers is 0. Show that the sum of their fourth powers when doubled yields a perfect square.

Let the three numbers be a, b, c; then $a + b + c = 0$, so $c = -(a + b)$. Hence:

$$\begin{aligned} &2\left(a^4 + b^4 + c^4\right) \\ &\quad = 2\left(a^4 + b^4 + (a + b)^4\right) \\ &\quad = 2\left(2a^4 + 2b^4 + 4a^3b + 6a^2b^2 + 4ab^4\right) \\ &\quad = 4\left(a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4\right) \\ &\quad = 4\left(a^2 + ab + b^2\right)^2. \end{aligned}$$

Hence the sum of the fourth powers when doubled equals the square of $2\left(a^2+ab+b^2\right)$.

Solution to problem IV-2-M.5

Consider these relations:

(1) $a-b-c=0$,

(2) $a^4+b^4+c^4=2\left(b^2c^2+c^2a^2+a^2b^2\right)$.

It is easy to prove (2) from (1) by simple manipulation. Now the interesting thing is: while identity (2) is symmetric in a,b,c, condition (1) is not so. How do you explain this?

The explanation lies in the full factorisation of the expression $P(a,b,c)$ given by:

$$P(a,b,c)=a^4+b^4+c^4-2\left(b^2c^2+c^2a^2+a^2b^2\right).$$

The information given in the problem implies that $a-b-c$ is a factor of $P(a,b,c)$. By symmetry, it follows that the following two expressions are factors as well: $b-c-a$, $c-a-b$. Hence $Q(a,b,c)=(a-b-c)(b-c-a)(c-a-b)$ is a divisor of $P(a,b,c)$.

Now observe the following: if we swap b and c in the expression $Q(a,b,c)$, two factors swap places while the third one remains the same, so the product of the factors remains the same. This tells us that if we were to expand the expression $(a-b-c)(b-c-a)(c-a-b)$, we would get an expression which is symmetric in a,b,c. And indeed we do:

$$(a-b-c)(b-c-a)(c-a-b)=a^3+b^3+c^3 \\ -\left(ab^2+a^2b+bc^2+b^2c+ca^2+c^2a\right),$$

which is symmetric as claimed.

What could be the fourth factor of $P(a,b,c)$? If we write

$$P(a,b,c)=(a-b-c)(b-c-a)(c-a-b) \\ \times \text{a fourth factor},$$

we see (by comparing degrees on both sides) that the fourth factor must be of degree 1. We also see that it must be completely symmetric in a,b,c; for it equals the ratio of two symmetric forms (P and Q) and hence is forced to be symmetric as well. These two conditions imply that it must be of the form $k(a+b+c)$ where k is some constant. By comparing the coefficient of a^4 on both sides, we quickly see that $k=1$. Hence:

$$P(a,b,c)=(a-b-c)(b-c-a)(c-a-b)(a+b+c).$$

So the desired explanation is this: though $a-b-c$ is not symmetric, it has companion factors, and together these factors make for a symmetric expression.

Solution to problem IV-2-M.6

Let a,b be two positive real numbers. Denote their product ab by P, and their sum $a+b$ by S. The following facts are known: if S is a constant, then the maximum value of P is $S^2/4$; and if P is a constant, then the minimum value of S is $2\sqrt{P}$. Use these results to find the maximum and minimum values taken by $x^2/\left(1+x^4\right)$.

Let $y=x^2/(1+x^4)$; then $1/y=x^2+1/x^2$. Since the product P of x^2 and $1/x^2$ is a constant ($P=1$), the sum of the two quantities is least when $x^2=1/x^2$, i.e., $x=\pm 1$, and the least value is 2. This means that the least value of $1/y$ is 2, hence the maximum value of y is $1/2$, taken when $x=\pm 1$.

For the minimum value: it is clear that $y\ge 0$, since only squared expressions and positive signs occur in the expression for y. And the value 0 is realizable, at $x=0$. Hence we have our result: $0\le y\le 1/2$. The lower bound is attained at $x=0$, and the upper bound at $x=1$.

Note. For the sake of completeness, we include a proof of the claim made in the statement of the problem: "If S is a constant, then the maximum value of P is $S^2/4$. If P is a constant, then the minimum value of S is $2\sqrt{P}$." Let the two quantities be a,b, and let $S=a+b$, $P=ab$. Invoking the following simple identity,

$$(a+b)^2-(a-b)^2=4ab,$$

we see that

$$4P=S^2-(a-b)^2, \quad S^2=4P+(a-b)^2.$$

From these relations, it follows that if S is a constant, then P will be largest when $a-b$ is least, i.e., when $a=b$. And if P is a constant, then S will be least when $a-b$ is least, i.e., when $a=b$. On substituting $a=b$ in the two relations, the two claims follow immediately.

Solution to problem IV-2-M.7

Given a parallelogram $ABCD$ and a point P inside the parallelogram such that $\measuredangle APB$ and $\measuredangle CPD$ are supplementary. Show that $\measuredangle PBC=\measuredangle PDC$.

There is an elegant pure geometry proof of the claim; it is illustrated in Figure 1. Translate the entire figure through the vector $\overrightarrow{BA}$; this maps A to A_1, B to A, C to D, D to D_1 and P to P_1. Each segment moves parallel to itself under the mapping. Hence $\measuredangle AP_1D=\measuredangle BPC$.

Now consider the quadrilateral AP_1DP. It is cyclic, since $\measuredangle APD+\measuredangle AP_1D=180°$. Hence $\measuredangle DPP_1=\measuredangle DAP_1$ ("angles in the same segment"). But $\measuredangle DPP_1=\measuredangle PDC$, since $PP_1 \parallel CD$, and $\measuredangle DAP_1=\measuredangle CBP$ by the nature of the translation. Hence $\measuredangle PDC=\measuredangle PBC$, as required.

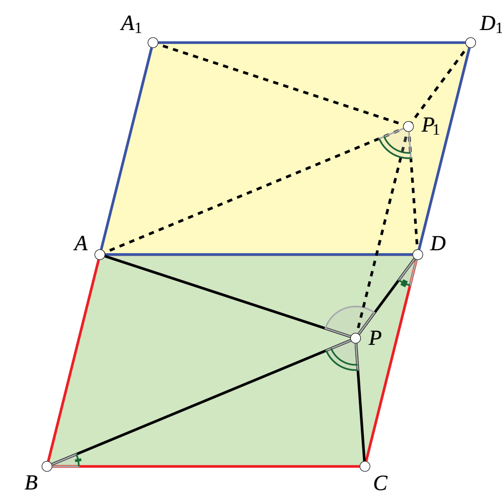


Figure 1.

MAKE THE RELATIONS COME OUT TRUE!

Insert math symbols between/before/after the digits on the left side to make each of the relations true.

- 0 0 0 = 6
- 1 1 1 = 6
- 2 2 2 = 6
- 3 3 3 = 6
- 4 4 4 = 6
- 5 5 5 = 6
- 6 6 6 = 6
- 7 7 7 = 6
- 8 8 8 = 6
- 9 9 9 = 6

Sanjay Nautiyal & Swati Sircar

problem corner

Problems for the Senior School

Problem Editors : PRITHWIJIT DE & SHAILESH SHIRALI

Problems for Solution

Problem IV-3-S.1
Determine all possible integral values of N such that $N(N-101)$ is the square of a positive integer.

Problem IV-3-S.2
Let R and S be two cubes with sides of lengths r and s, respectively, where r and s are positive integers. Show that the difference of their volumes numerically equals the difference of their surface areas if and only if $r = s$.

Problem IV-3-S.3
Suppose $S = \{0, 1\}$ has the following addition and multiplication rules:

$$0+0=0, \quad 0+1=1+0=1, \quad 1+1=0,$$
$$0\times 0 = 1\times 0 = 0\times 1 = 0, \quad 1\times 1 = 1.$$

(The rules will make more sense if you think of 0 as representing Even, and 1 as representing Odd. For example, the sum of two Odd numbers is Even, and the product of two Odd numbers is Odd.)

A system of polynomials is defined with coefficients in S. The sum and product of two polynomials in the system are the usual sum and product, respectively, where for the addition and multiplication of coefficients the above rules of addition and multiplication apply. For example:

$$\begin{aligned}(x+1)\times\left(x^2+x+1\right) &= x^3+(1+1)x^2+(1+1)x+1\\ &= x^3+0x^2+0x+1 = x^3+1.\end{aligned}$$

Show that in this system x^3+x+1 is not factorisable, that is, one cannot write

$$x^3+x+1 = (ax+b)\times(cx^2+dx+e),$$

where $a, b, c, d, e \in S$.

Problem IV-3-S.4
Consider all non-empty subsets of the set $\{1, 2, 3, \dots, n\}$. For each such subset, find the product of the reciprocals of each of its elements. Denote the sum of all these products by a_n. For example,

$$\begin{aligned}a_3 &= \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{1\times 2}+\frac{1}{1\times 3}+\frac{1}{2\times 3}\\ &\quad+\frac{1}{1\times 2\times 3}\\ &= \frac{6+3+2+3+2+1+1}{6} = 3.\end{aligned}$$

Prove that $a_n = n$ for all positive integers n.

Problem IV-3-S.5
Show that the polynomial $x^8-x^7+x^2-x+15$ has no real zero.

Solutions of Problems in Issue-IV-2 (July 2015)

Solution to problem IV-2-S.1
Starting with any three-digit number n we obtain a new number $f(n)$ which is equal to the sum of the three digits of n, their three products in pairs and the product of all three digits. (Example: $f(325) = (3+2+5)+(6+15+10)+30 = 71$.) Find all three-digit numbers such that $f(n) = n$. [Adapted from British Mathematical Olympiad, 1994]

Let $n = 100a+10b+c$ where $a, b, c \in \{0, 1, \dots, 9\}$, $a \neq 0$. Then $f(n) = a+b+c+ab+bc+ca+abc$, so if $f(n) = n$ then we must have

$$99a+9b = ab+bc+ca+abc. \qquad (1)$$

This leads to

$$a(99-b-c-bc) = b(c-9). \qquad (2)$$

Observe that $99-b-c-bc \geq 0$ and $c-9 \leq 0$. Thus equation (2) holds if and only if $c = 9$ and $b+c+bc = 99$, and the latter relation shows that $b = 9$ as well. But when $b = c = 9$, both sides of (1) reduce to $99a+81$, and the equation reduces to an identity. Thus a is not unique but can take any value between 1 and 9. Hence the three-digit numbers n satisfying $f(n) = n$ are 199, 299, 399, 499, 599, 699, 799, 899 and 999.

Solution to problem IV-2-S.2
Solve in integers the equation: $x+y = x^2-xy+y^2$.

The given equation can be written as:

$$(x-y)^2+(x-1)^2+(y-1)^2 = 2. \qquad (3)$$

If $x = y$ then $|x-1| = |y-1| = 1$, whence $(x, y) = (0, 0), (2, 2)$. If $x \neq y$ then $|x-y| = 1$ and either $x = 1$ or $y = 1$. If $x = 1$ then $y = 0$ or $y = 2$. Thus the solutions are $(x, y) = (1, 0), (1, 2)$. Since (y_0, x_0) is also a solution if (x_0, y_0) is a solution, $(x, y) = (0, 1), (2, 1)$ are also integer solutions of the equation. Thus the integer solutions are $(x, y) = (0, 0), (0, 1), (1, 0), (1, 2), (2, 1), (2, 2)$.

Solution to problem IV-2-S.3
Let a, b, c be the lengths of the sides of a scalene triangle and A, B, C be the opposite angles. Prove that

$$\begin{aligned}2(Aa+Bb+Cc) &> Ab+Ac+Ba+Bc\\ &\quad+Ca+Cb.\end{aligned}$$

Without loss of generality, assume that $a > b > c$. Then $A > B > C$. Now:

$$\begin{aligned}&2(Aa+Bb+Cc)\\ &\quad-(Ab+Ac+Ba+Bc+Ca+Cb)\\ &\quad= (A-B)(a-b)+(A-C)(a-c)\\ &\quad\quad+(B-C)(b-c).\end{aligned}$$

Clearly the right-hand side is positive.

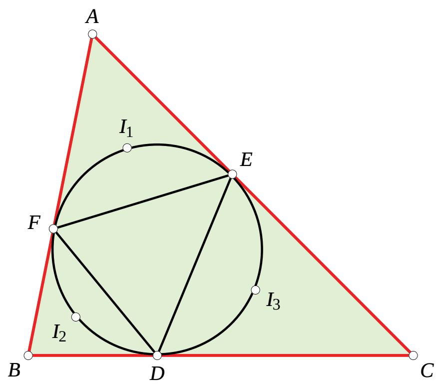
Solution to problem IV-2-S.4
Three positive real numbers a, b, c are such that

$$a^2+5b^2+4c^2-4ab-4bc = 0.$$

Can a, b, c be the lengths of the sides of a triangle? Justify your answer. [Regional Mathematical Olympiad, 2014]

No. Note that $a^2+5b^2+4c^2-4ab-4bc = (a-2b)^2+(b-2c)^2$, hence the stated condition implies that $(a-2b)^2+(b-2c)^2 = 0$, which in turn implies that $a:b:c = 4:2:1$, i.e., $b+c:a = 3:4$. But this violates the triangle inequality, for $b+c$ must exceed a (strictly).

Solution to problem IV-2-S.5
Let D, E, F be the points of contact of the incircle of an acute-angled triangle ABC with the sides BC, CA, AB, respectively. Let I_1, I_2, I_3 be the incentres of the triangles AFE, BDF, CED, respectively. Prove that the lines I_1D, I_2E, I_3F concur. [Adapted from the Regional Mathematical Olympiad, 2014]



Observe that $\measuredangle AFE = \measuredangle AEF = 90^\circ - \frac{1}{2}A$ and $\measuredangle FDE = \measuredangle AEF = 90^\circ - \frac{1}{2}A$. Again, $\measuredangle EI_1F = 90^\circ + \frac{1}{2}A$. Hence:

$$\measuredangle EI_1F + \measuredangle FDE = 180^\circ,$$

so I_1 lies on the incircle. Also:

$$\measuredangle I_1FE = \frac{1}{2}\measuredangle AFE = \frac{1}{2}\measuredangle AEF = \measuredangle I_1EF.$$

Thus $I_1E = I_1F$. But then they are equal chords of a circle, so they subtend equal angles at the circumference. Therefore $\measuredangle I_1DF = \measuredangle I_1DE$ and so I_1D is the internal bisector of $\measuredangle FDE$. Similarly we can show that I_2E and I_3F are internal bisectors of $\measuredangle DEF$ and $\measuredangle DFE$, respectively. Thus the three lines I_1D, I_2E, I_3F are concurrent at the incentre of triangle DEF.

ICMI AWARDS 2015

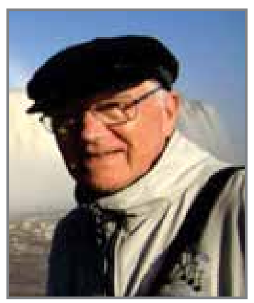
The International Committee for Mathematics Instruction (ICMI) has just announced the recipients of the 2015 Felix Klein and Hans Freudenthal Awards:

- **Felix Klein Medal:** Professor Alan J. Bishop, Emeritus Professor of Education at Monash University, Melbourne.
- **Hans Freudenthal Medal:** Professor Jill Adler, Chair of Mathematics Education, University of the Witwatersrand, South Africa.

The medals will be awarded in the opening ceremony of the International Congress for Mathematics Education (ICME-13) to be held in July 2016 in Hamburg.

Here are brief summaries excerpted from the official citations released by the Awards Committee. See the website http://www.icmihistory.unito.it/ to get a sense of the history of the ICMI and a sense of the work done by Felix Klein and by Hans Freudenthal.

ALAN J. BISHOP



The **Felix Klein medal** is awarded for life-time achievement in mathematics education research. [It] is aimed at acknowledging scholars who have shaped our field over their lifetimes. Past candidates have been influential and have had an impact both at the national level and the international level. [They have] made substantial research contributions, introduced new issues, ideas and perspectives. Additional considerations have included leadership roles, mentoring, and peer recognition.

Alan J. Bishop, Emeritus Professor of Monash University, Australia is the awardee for 2015. He has been instrumental in bringing the political, social, and cultural dimensions of mathematics education to the attention of the field. His early research was on spatial abilities and visualization, but later he worked on the process of mathematical enculturation and how it is carried out in different countries. Subsequently he developed the notion of mathematics as a cultural product. Over more than 45 years of sustained, consistent work, this led to a great deal of work on the political and social dimensions of mathematics education.

Alan Bishop served as editor of *Educational Studies in Mathematics* from 1979 to 1989. In 1980, he founded and became the series editor of Kluwer's *Mathematics Education Library*. He served as the chief editor for many editions of the *International Handbook of Mathematics Education*. Through his tireless work in the area of publication, he enabled research in mathematics education to become an established field.

His education was at Harvard and later at Hull. After a stint at Cambridge University, he moved to Monash University, Australia. Through the Association of Teachers of Mathematics, he worked as a mentor to numerous teachers and supervised many doctoral students, several of whom became distinguished internationally. Through his work in forging links between research and practice, he helped mathematics educators establish communities of inquiry by teaching courses, speaking at conferences and workshops, directing research and development projects, and serving as a consultant, a project evaluator, and an external examiner.

As noted by one of the nominators, "Alan is an excellent scholar and researcher who shaped our field not only over his lifetime but also over its lifetime, not only in England and Australia ... but also internationally."

review

Review of
The Sand Reckoner
by Gillian Bradshaw[1]

'There was a world there, a world without material existence but luminous with pure reason, and they couldn't see it!'

Dakshayini Suresh

Archimedes (Syracuse, 287 BC-212 BC) is generally believed to have been the greatest mathematician of antiquity, and certainly one of the three greatest of all time (along with Newton and Gauss). He is probably known best for his articulation of what has come to be known as the Archimedes principle, or rather for the entertaining scene that is said to have ensued upon its discovery. The story goes as follows.

Archimedes was asked by King Hieron of Syracuse to determine whether a gold wreath he had commissioned and subsequently received was, in fact, silver. While turning this problem over in his mind, Archimedes chanced to go for a bath, and it struck him, as he bathed, that the volume of water displaced by his being in the bath was equal to the volume of his own body. When he made this discovery, he is said to have run straight out of the bath and his home naked, shouting 'Eureka, eureka!' ('I have found it!'). He used this rule of displacement to determine whether the crown actually was pure and weighed as much as a pure gold object of the same volume.

[1] *The title of the novel is the name of a treatise by Archimedes in which he sets out to estimate the number of grains of sand it would take to fill the universe. In the opening scene of the novel, Bradshaw's Archimedes is trying to imagine values large enough to express the number of grains of sand in a box. In the novel, Archimedes also designs a catapult that comes to be known as the Sand Reckoner.*

Keywords: *Archimedes, Syracuse, Eureka*

He found that the crown was not made of pure gold and that his king had indeed been cheated.

The *Sand Reckoner* is a historical novel set in Ancient Greece. The young Archimedes is twenty three years old. He has just returned from an exciting and intellectually productive hiatus in Alexandria to find that his father is dying. His family needs an income. They need to sell off one or two of their four slaves. They need the only able male member of their family to find employment.

For about 300 years, during the reign of the Ptolemies, Alexandria was a city largely at peace within itself and with the rest of the world. In this period, it attracted many scholars who came to study and teach in the University of Alexandria. In Archimedes' circles there were scholars who had worked alongside Euclid. Education in Alexandria was certainly a rich opportunity for intellectual growth and exposure to the ideas and methods of the era.

Archimedes himself seems to want nothing more than to draw circles in the sand, to experiment with his abacus. Meals, finances, these things are not important to him. The job of accounting for the responsibilities and monetary concerns of his life falls to his slave Marcus. Finding the household in disarray comes as a rude shock to Archimedes. He must do his duty as a son. From this point onwards, Bradshaw unfurls the many intertwining plot lines that build the book. A diverse cast of characters appears: Archimedes' parents, affectionate, aged; his sister, lovely, supportive, but strangely sidelined; his lady love, the daughter of the reigning Tyrant Hieron, as fascinated with music and its governing technicalities as Archimedes himself is; the competitive and jealous engineers vying for his job. He wins the Regent over with his engineering skills and secures a position as the Tyrant's chief engineer.

The method by which he is said to have won favour with his king was by proving how his use of levers and pulleys could move an object of any weight. He chose to demonstrate this on a large ship, complete with crew and cargo. By operating his pulleys such that the ship did indeed move across the water without any great effort, he secured a position as the king's chief engineer of war machines.

Such is the progression of Archimedes' career, as described by Bradshaw. However, the most complex character in the book is probably Marcus, the slave, and all other plot lines apart, he deserves special mention. He is everything his master is not—decisive, headstrong, emotional and practical—and is caught in the drama of existence, the problems posed by the world, not in the problems of mathematics that Archimedes invents for pleasure. From the outset, it is clear that his attitude towards his master's family and towards Syracuse is not one of servility and fierce loyalty, but rather one of grudging attachment, formed over his years of residence and service. It soon becomes clear that Marcus' being of uncertain descent and his having slipped into Sicily quietly after a war are of more significance than they seem. Syracuse is at war with Rome, and Marcus' identity becomes a point of contention. If he is indeed Roman, as some of the Greek soldiers suspect, then his loyalty, which has till now been taken for granted, will come into question. His having been born free, rather than into a life of slavery, also means in some subtle way that his relationships with those he serves are sometimes ambiguous. Hence, when he returns from Alexandria with Archimedes, he begins a somewhat fraught flirtation with the daughter of the house, Philyra, now a young lady in her own right. It is not long before he realises that the ongoing war is a serious threat to his life, and that without him, the balance of his young master's life will be compromised. Even though *The Sand Reckoner* is technically about Archimedes, the nuanced and tragic character of Marcus is perhaps more engaging. He possesses a sense of spirit and volition which his master seems to lack.

Archimedes for a long time has very little inkling of the concerns that plague his friends and family. He has a lot of affection for people, but often struggles to express it. The social world puzzles him. Bradshaw successfully conveys that there is something extraordinary about her protagonist aside from his withdrawn manners and daydreaming tendencies. She illustrates what is different about him without embarking on too many complicated, technical math-monologues. She focuses on his wonder and his skill for innovating in a way that makes his brilliance accessible to all readers. Archimedes makes his living building weapons for the king. He offers to engineer catapults and is deeply confident of his ability to manufacture rare machines (one-talenters and two-talenters, catapults that can throw ammunition that weighs as much as one or two men). Rather, he is confident of the mathematical logic behind the construction of these catapults. He explains, and successfully demonstrates that increasing the dimensions of the parts of the catapult while maintaining the ratios of the parts to one another is the key to creating a large weapon. In his mind it is a matter of imagining and calculating. And in the minds of those around him, he has surpassed the limits of what is imaginable.

Archimedes is generally believed to have been happier in the world of theories and abstraction than in the realm of mechanics and concrete application. Plutarch writes that he possessed a 'lofty' spirit and a 'profound' soul. Indeed, most stories about Archimedes portray him as distracted from his physical surroundings (running naked through the streets, for instance) and intent on his mathematics, to the point of putting his own life in danger. (One version of the story of his death says that he provoked a soldier by refusing to obey orders because he was engrossed in solving a mathematical problem.) However, it is the same Plutarch who describes the efficiency and cruelty of the war machines that Archimedes designed to protect Syracuse—machines responsible for the slaughter of thousands of Roman soldiers.

Archimedes finds that for him, the magic of mathematics and mechanics lies in what is new and yet unchartered territory. He invents, and enjoys the process, but producing catapults or pumps (water snails as they come later to be called, because of the spiral tube of reed that enables water to flow uphill and downhill) bores him, and he loses interest once the job becomes repetitive. The author portrays her protagonist as a man whose mind prefers to 'soar' and dream and to attempt to know and make *new* things. He is only grounded by the rules and precision of his subject; no other limits seem to matter.

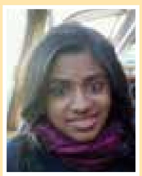
However, the novel dwells more on his growth as a person than as a mathematician. Through the course of the plot, the young Archimedes discovers that the world is a more practical and brutal place than he imagined, and that limits exist in society, whether or not he wants them to exist in his mind. He faces many confusing challenges: He engages in a wholly impractical romance with the princess Delia, he watches his once brilliant father wither into someone unrecognisable. His friends are endangered. Worst of all, he faces the consequence of his brilliance. However interesting the process of designing a catapult may be, he comes to see that the effect of such a weapon is basically massacre. And no sense of loyalty, or duty, or love for theoretical and concrete wonders of mathematics can change this.

Ultimately, the novel is about much more than Archimedes. It is about the contrast between the inner world of the mind and the more pressing world of real life demands and constraints that exist around us. In this realm duty, loyalty, marriage, war and death take precedence over the power of imagination and logic. Hence, Gillian Bradshaw's Archimedes seems to use his mind as refuge from all that is disorderly in the external world. His mathematics is his safe haven, a space that remains neat, unsullied by all the pain and tragedy of life.

There are several retellings of the story of Archimedes' death. In all of them, he dies at the hands of a Roman solider, after Syracuse has been defeated. In many, he dies merely because he provokes this soldier without intending to, by carrying measuring instruments that are mistaken for treasure, or by refusing to follow the ranks for prisoners when ordered to, or just by not realising that his city had fallen to the Romans, because he was too busy drawing figures in the sand. In nearly every version, Marcellus, the Roman general, sorely regrets the loss of this brilliant mind.

The Sand Reckoner is an enjoyable read for anyone fifteen and up, with basic knowledge of tenth standard mathematics. It may be easier to appreciate this novel if one is already immersed in the world of math and math history, but it will certainly also appeal to readers whose knowledge of the subject does not go beyond the very basic.

But the charm of the book is that it will convey the same sense of awe and excitement to everyone. It will place mathematical discovery and its applications in a historical and social context. It is the ideal way to illustrate the story-like quality of the course of math history to even the most reluctant and intimidated disciples of the subject.



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... Awards continued from Page 94

ICMI AWARDS 2015

JILL ADLER

The **Hans Freudenthal medal** is aimed at acknowledging the outstanding contributions of an individual's theoretically well-conceived and highly coherent research programme. It honours a scholar who has initiated a new research programme and has brought it to maturation over the past 10 years. The research programme is one that has had an impact on our community. It is also intended that a Freudenthal awardee should still have a minimum of a decade of active research work ahead of him or her so as to continue contributing to the field. In brief, the criteria for this award are depth, novelty, sustainability, and impact of the research programme.

Professor Jill Adler, FRF Chair of Mathematics Education, University of the Witwatersrand, South Africa is the awardee for 2015, in recognition of her outstanding research program dedicated to improving the teaching and learning of mathematics in South Africa – from her 1990s ground-breaking research on the dilemmas of teaching mathematics in multilingual classrooms, to her subsequent focus on problems related to mathematical knowledge for teaching and professional development. Her research has served to advance understanding of the relationship between language and mathematics in the classroom.

Over the last two decades, she spearheaded several large-scale teacher development projects aimed at developing mathematics teaching practice at the secondary level, so as to enable more learners from disadvantaged communities qualify for entry to mathematics-related courses at university.

Jill Adler was born in Johannesburg and graduated from the University of the Witwatersrand with a B.Sc. in mathematics and psychology (1972). She taught secondary school mathematics in a so-called 'coloured' school in Cape Town – an experience that she credits for strengthening her concerns about educational inequality and leading her to work in that direction. This was followed by many years spent on developing materials for adults and alienated youth excluded from school mathematics learning in apartheid South Africa. In 1985, she obtained a M.Ed. for her dissertation: *Mathematics by newspaper in South Africa: junior secondary mathematics for adults through the medium of a newspaper. Her doctoral research (1996) was titled: Secondary teachers' knowledge of the dynamics of teaching and learning mathematics in multilingual classrooms.*

In addition to her international research at the cutting edge of the field, she has played an outstanding leadership role in mathematics education research in South Africa, Africa, and beyond, and has helped in adding to the human research capacity in Southern Africa. Her contributions to the development of research and practice have earned her leadership positions in renowned international and national professional associations.

A Review of Math! Encounters with High School Students

Dialogue and Mathematics—Serge Lang Style!

Shashidhar Jagadeeshan

The notion of dialogue and mathematics may at first seem a strange combination, but if one thinks about it, often in a lively interactive classroom this is exactly what is transpiring. According to the late physicist David Bohm, the root of the word *dialogue* comes from the Greek *dialogus*. The word *logos* in turn can be interpreted as 'meaning of the word' and *dia* means 'through'. So dialogue can then be seen as a process where there is a flow of meaning. All teachers would agree that this is what they would like in their classrooms.

The book under review, *Math! Encounters with High School Students* by Serge Lang, is an old one, published in 1985, but well worth bringing to the notice of students and teachers of mathematics. It is a series of seven dialogues on mathematics with school students and a postscript discussing mathematics teaching.

Apart from the content, which I will discuss later, the book is unique in its style of delivery. Even though we are not in the audience, we can sense the energy and excitement of the exchange. One wonders (without being completely reductionist), what are the ingredients needed for such a flow of energy and meaning to take place between teacher and taught? The obvious

Keywords: *Dialogue, facilitation, pedagogy, creativity*

ones are a mastery of the subject on the part of the teacher, an ability to gauge the level of the students and begin from where they are, a sense of humour, encouraging students to think on their feet, generating a creative tension and finally pulling it all off.

The excerpts below illustrate these points well.

Excerpts from page 20

Serge Lang: . . . $2\pi r = C$. There is your formula. Do you agree that's a proof? [*Serge Lang points to Rachel.*]

Rachel: Yes. [*Her tone is uncertain.*]

Serge Lang: You do?

Rachel: Yes. [*Laughing a little.*]

Serge Lang: What do you mean 'yes'? Is it yes by intimidation or a yes by conviction? Or a little bit of both?

Rachel: A little bit of both. [*Laughter.*]

Serge Lang: Well, where is the intimidation?

Rachel: I don't know.

Serge Lang: You don't know? [*Laughter.*] All right, let's make it all conviction. Look, where do I start from? . . .

Excerpts from pages 34 and 35

Serge Lang: . . .Do you accept all that? [*Students approve . . .*] So we can make a general result:

Under dilation by a factor of *r*, *s*, *t* in the three dimensions, the volume of a solid changes by the factor of the product, *rst*.

Just like yesterday: area changes by a factor of r^2 if we dilate by r in each direction; a factor of rs if we dilate by a factor of r in one dimension and s in the other dimension; and now volume changes by a factor of rst if you dilate by a factor of r in one dimension, s in the another and t in the third.

And the three dimensions are in perpendicular directions. Now I will deal mostly in three dimensions, but what would be a natural generalization of this? Serge.

Serge: (a student): I don't know.

Serge Lang: What's a generalization of what I have just done there? I started in 2 dimensions, then I went to 3 dimensions . . . .

Serge: [*Interrupts.*] Four dimensions. OK. It's the next product. I see. It's *rst* whatever.

Serge Lang: Ah, *rst* whatever. That's right. So suppose I have a solid in four dimensions. You see the four dimensions? Now I can't draw it.

Serge: Well, you could not draw it either in three dimensions!

Serge Lang: That's a very good remark. You are absolutely right. So the truth of what I am saying does not depend on my ability to draw the picture! . . .

And suppose I have a solid in n dimensions, and I make the dilation by factor of r in all directions, in all n dimensions. How does the volume change?

Serge: r to the power n.

Serge Lang: r^n, and that's how it is in n dimensions. OK? Any problems? Sandra.

Sandra: No. [*The other students nod, and seem perfectly at ease.*]

Serge Lang: . . .But I think it's remarkable how you react to the possibility in n dimensions.

[*Laughter.*] I am slightly taken aback at the way you just went along with it.

Excerpts from page 120

Student: So far you used different methods; you first used one method, then you changed the method; probably a different method would do it for all numbers.

Serge Lang: That is a very weak argument. [*Laughter.*] Because the argument is based on psychology, and I am asking you to deal with mathematical problems. Not psychological ones. [*Laughter.*] So if you start basing your mathematical intuition on my psychology, [*Laughter.*] you're going to have a hard time with it. That's dangerous. Think again.

[*Students talk among each other.*]

Serge Lang was born in 1927 in Paris and died in 2005 in Berkeley, California. Anyone who has studied higher mathematics would be familiar with his name as the author of mathematics books on almost every topic under the sun! On his retirement from Yale University in 2005, where he was a faculty member from 1972, Yale president Richard C. Levin shares a joke about this. "Someone calls the Yale Mathematics Department, and asks for Serge Lang. The assistant who answers says, 'He can't talk now, he is writing a book. I will put you on hold'." He was a prodigious author and wrote more than 61 books (some feel this is an underestimate) and 120 research articles. Most famous amongst his textbooks is *Algebra*, which is a classic in the area. For school teachers, apart from the book under review, I would recommend they refer to [2] and [3].

Lang could not have had a better mathematical lineage. He wrote his PhD thesis under the famous algebraist Emil Artin and did postdoctoral work with André Weil. He won the Cole Prize (1959) and the Steele Prize (1999) of the American Mathematical Society. He was elected to the National Academy of Sciences in 1985.

He was a deeply committed teacher with a great passion for communicating mathematics and devoted a considerable part of his life to teaching. In recognition for his commitment he was awarded the Dylon Hixon Prize for teaching in Yale College. His passions included mathematics, music and politics ('trouble making' in Lang's words). Jorgenson and Krantz pay him the greatest compliment (from my point of view) that a person can receive: "Serge Lang's greatest passion in life was learning" [1]. He demonstrated this by writing books and teaching courses in new areas of mathematics, because he believed that that was the best way to learn. He was famous for cajoling young mathematicians to teach him new mathematics.

Reading the article [1] on Lang by Jorgenson and Krantz, where several famous mathematicians recall their interactions with him, a picture emerges of an extremely colourful and energetic personality, not always the easiest of persons to relate to. Any attempt to categorize him would soon fail, because he seems to be rather short tempered and confrontational, but at the same time kind and generous, especially to young people and his students. He was driven by strong convictions and fought several public battles based on these convictions. It is best to quote Lang on this!

I personally prefer to live in a society where people do think independently and clearly. One of my principal goals is therefore to make people think. When faced with persons who fudge the issues, or cover up, or attempt to rewrite history, the process of clarifying the issues does lead to confrontation, it creates tension, and it may be interpreted as carrying out a 'personal vendetta' . . . . I regard such an interpretation as very unfortunate, and I reject it totally.

Let us turn our attention to the contents of the book. The intention is to make beautiful mathematics accessible to students of roughly class 8 to 10. The first dialogue is called "What is *pi*?" It is extremely important that high school students get a good understanding of this well-known constant of nature. The misconceptions about π that I encounter among teachers and students often alarm me! They remember it as the fraction $\frac{22}{7}$, or 3.14, and very few are aware that it is irrational, let alone transcendental. Lang actually deals with the subtle point as to why the same constant π shows up both in the formula for the circumference and area of a circle.

Dialogues 2 to 5 deal with derivations of the formulae for the volume of a pyramid, cone and sphere and the formulae for the circumference of the circle and the surface area of the sphere. Lang uses essentially Archimedes' method of 'exhaustion' for these derivations. As far as I am aware, standard school mathematics textbooks rarely derive these formulas. Perhaps there is a feeling that these derivations are too difficult, or that they are best done using integral calculus. But, as Lang so aptly demonstrates, they are very accessible to younger students, and in fact if done before the students see integral calculus, it serves to show not only the power of calculus, but also the limitations of the method of exhaustion.

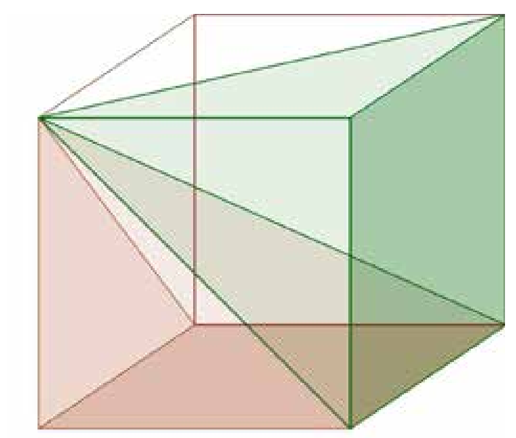

Figure 1.

One of my favourite parts is the derivation of the volume of a pyramid. Lang and his students stumble upon the special case of the cube, which can actually be divided into three congruent pyramids (see Figure 1). This helps us to understand where the $\frac{1}{3}$ factor comes in. It is also a nice activity to get students to make nets of these pyramids, as many surprises await the student in doing so!

Dialogue 6 deals with Pythagorean triplets. Here students are introduced to the problem and the complete solution is demonstrated with the help of the parametric representation of the unit circle

$$x(t) = \frac{1-t^2}{1+t^2} \text{ and } y(t) = \frac{2t}{1+t^2},$$

explaining the geometric significance of the parameter t. Here t is the slope of a special line, and one gets a very elegant connection to double angle formulae from trigonometry.

The last mathematical dialogue deals with infinities, taking students from the very basics all the way to the result that real numbers are not denumerable.

I have used this little book in many ways as a teacher: as a reference book, as a model to conduct mathematical dialogues and also a source for students to read on their own and make presentations. In short, I would highly recommend it to students and teachers of secondary and high school.

I would like to end the review with comments and excerpts illustrating Lang's views on mathematics education from the preface and from the Postscript, which is also a dialogue among Lang, educators and a student. Lang has strong views on the curriculum: to quote him from the preface,

A lot of the curriculum of elementary and high schools is very dry. You may never have had a chance to see what beautiful mathematics is like. . . . I have many objections to the high school curriculum. Perhaps the main one is the incoherence of what is done there, the lack of sweep, the little exercises that don't mean anything.

In reaction to the feeling that school students are not mature enough to see proofs,

There is the scandal! Those proofs are very beautiful, it's real mathematics. They allow you to appreciate mathematics, to see why something is true by using arguments which are quite understandable.

But in the course of the same dialogue, almost contradicting himself, he insists that memorization of formula is essential!

There is no way to avoid this, so you must ask kids to repeat the formula ten times. . . . It must be driven into their ears like music. You shouldn't ask every time why the formula is true.

One may not agree with Lang's philosophy or ideas all the time, but he does force you to think about what we are doing as teachers. He ends the dialogue on a more humane note and we will leave the readers with that.

Each teacher must do according to his own way, his own taste. Each one must use their own means to excite the students. One needs everything without exclusivity.

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REVERSIBLE PRIMES

A **reversible prime** is one which when one reverses the order of its digits remains a prime number. Examples: the primes 13 and 17. Obviously, the definition is base dependent; here it is assumed that we are working in base 10 (the decimal system). A question of interest: *How common are such primes?* The table below gives the relevant data. We use the following notation: $f(n)$ is the number of n-digit primes, and $g(n)$ is the number of reversible n-digit primes. Note that $f(1) = g(1)$, since any single-digit prime trivially satisfies the definition of reversibility.

n	$f(n)$	$g(n)$	Examples of reversible n-digit primes
1	4	4	2, 3, 5, 7
2	21	9	11, 13, 17, 31, 37, 71, 73, 79, 97
3	143	43	101, 107, 113, 131, 149, 151, 157, 167, . . .
4	1061	204	1009, 1021, 1031, 1033, 1061, 1069, . . .
5	8363	1499	10007, 10009, 10039, 10061, 10067, . . .
6	68906	9538	100049, 100129, 100183, 100267, 100271, . . .
7	586081	71142	1000033, 1000037, 1000039, 1000117, 1000159, . . .

We see that reversible primes occur rather more frequently than one may expect! For example, among the four-digit primes, close to 20 percent of them are reversible.

We close with an interesting question: how does the fraction of primes that are reversible change with the number of digits? From the data it is evident that the fraction slowly decreases with n. However, the exact nature of this decrease is difficult to predict. Clearly, a more detailed study will be required to ascertain this.

The Closing Bracket . . .

A wide spectrum of educationists in this country complain that the number and quality of people engaged in mathematical careers, from researchers to educators at all levels, is woefully inadequate. This is seen in a lacuna of competent research mathematicians, difficulty in filling positions in mathematics departments and the poor quality of teaching in schools and colleges.

In a vast country like ours, independent for less than 100 years, with a huge population and a great struggle for distribution of resources in a sustainable and equitable manner, there are obviously a variety of reasons why this is so.

I would like to speculate on the negative impact of high-stakes entrance exams on mathematics education and the potential for interest in pursuing a career in mathematics. I say speculate, because to prove my point one would have to undertake a proper survey and gather enough data to prove or disprove my hypothesis! But, if the readers will indulge me a bit, I would like to at least state my concerns.

The majority of Indian children who survive high school will not encounter excellence in mathematics education, because of a lack of resources and good teachers, and one may be tempted to say that this is the reason that there are not enough competent professionals in mathematics. Granted that this is a major factor, but if one thinks about it, there is still a sizable urban population of students who have access to quality education. For this fortunate minority, resources and competency of teachers is not an issue. In fact, looking at the problems that students in India need to solve in many high-stakes entrance exams, I would take a safe bet that they are much 'harder' than the ones many other students in the world will encounter. Yet, a very small fraction of these students take up careers in mathematics. An obvious reason offered is that these careers are not lucrative. However, this is true the world over, and yet many countries do not suffer from such a severe shortage.

I would like to suggest that one of the major unexamined reasons for this shortage is that mathematics has become a gatekeeper in accessing quality higher education. The problems that students need to solve in these exams are highly contrived and offer no depth of understanding in mathematics. Entrance exams have become an end in themselves, and in order to 'crack' them, students have to perform several years of 'tapas': sacrificing their precious childhood to solve hundreds of problems, learn inane tricks, and memorize dozens of identities. Very often, they have to learn advanced topics not routinely covered in high school syllabi, in order to acquire an edge over their competitors. All this has to be done under extremely tight conditions of time and also penalisation for mistakes.

What is the long-term impact of such a system? Many have commented on its negative psychological effects on young people, but here let us look at its effect on mathematics education alone. The impression that students have of mathematics is that it requires a lot of gymnastics, that it makes the difference between success and failure, and that it is accessible only to the few. They do not view mathematics as an area of great depth and beauty, a potential source of joy and enjoyment throughout one's life, or as a serious career option at many levels. Obviously those who perform poorly on the entrance exams develop an allergy to mathematics. Some who fail to enter professional streams, and therefore pursue an undergraduate education in mathematics, are there not out of choice; having failed to clear the bar, they are often diffident about their own abilities. What about the successful stories on the high-stakes exams? Colleagues who teach at premier institutions whose admission policy is based on these entrance exams frequently complain that their students are 'burnt out' after so many years of hard work. They are often reluctant to learn new mathematics, because they have a false impression that they have seen it all, and know it all! Very few of them go on to pursue mathematics for its own sake, and this tiny fraction often ends up migrating to countries with more conducive overall work environments.

What do we do about this dismal state of affairs? The problem seems rather large and perhaps we teachers feel a bit helpless given the enormity of the challenge. To hope for a change at higher levels of policy-making may be very optimistic, but is there anything that we can do that begins in our class rooms? I think, first and foremost, we must understand that mathematics needs a variety of practitioners so that it can flourish: researchers, teachers, amateurs, writers and historians. More important, we must realize that that we need a culture of enjoying mathematics rather than using mathematical aptitude (dubiously defined and measured) to distribute resources amongst its people. It is only such a culture that is sustainable and that will have a lasting impact on the mathematical scene in India.

– Shashidhar Jagadeeshan

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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Rishi Valley

TEACHING Thinking Skills

PADMAPRIYA SHIRALI

A PRACTICAL APPROACH

At Right Angles
A Resource for School Mathematics

Thinking Skills

Practically every human activity involves usage of thinking skills. What are thinking skills? They are essentially mental processes that we do: classifying objects, observing properties, encoding information, comparing, taking decisions, making inferences and solving problems. Thinking skills can be viewed as the building blocks of the whole canvas of thinking. These thinking skills are broadly classified into two categories: *lower order thinking skills and higher order thinking skills.*

In the context of mathematics teaching in many of our schools, we tend to focus more on lower order thinking skills and do not pay sufficient attention to higher order thinking skills. For instance we tend to focus more on recall of information like multiplication facts, computational skills, procedures, formulae and definitions. We do not pose enough problems which require students to identify relationships and patterns, establish connections, approach a problem in different ways, make inferences and predict outcomes, generate new questions or explorations, generalize, etc. Also, many of our textbooks do not lend themselves to the teaching of these higher order thinking skills. Most problems are procedure oriented and repetitive; they can be solved in a mechanical fashion. There is very little scope for reasoning, investigating, discovering, predicting. Nor is there any scope for challenge and creativity. Children need exposure to problems requiring higher order thinking skills. All children deserve such experiences - the challenge and enjoyment of interesting problems in mathematics.

Children who have developed these skills see connections, patterns and structures in a variety of situations. They are able to generalize these patterns and make statements about them. They are able to organize and categorize information. They think symbolically and logically with quantitative and spatial relations.

How does a teacher create opportunities for children to build and use these higher order thinking skills? They will need to identify a set of these skills, select problems which lend themselves to the usage of these skills and pose questions which will help the child in developing various such skills.

The focus of this article is on developing a subset of thinking skills some of which are related to the topics covered in the primary school but go much beyond; in the process, they deepen the child's understanding of concepts and help in appreciating logic and order inherent in mathematical thinking. I have selected a set of problems which I have used during the course of my own teaching. They require varied thinking skills involving number manipulation, geometric visualisation, logical thinking and experimentation. The given material can be approached at many levels of thinking. Skills and strategies which work in one situation may not work in another.

Keywords: *Thinking skills, logic, reasoning, exploring, pattern, conjecture*

ACTIVITIES

Some Guidelines for Teachers in using these problems

- **Tasks:** Most of the tasks are accessible to all at the start. Many are extendable and lead to further challenges. Let children search in various directions. It is important that they are allowed to figure out the solutions in their own way. By explaining these problems a teacher can ruin the pleasure of discovery and insight.
- **Time:** Give children plenty of time to solve these problems. Do not rush them. Some problems can be attempted by individual students. Some can be attempted in pairs. A group of four students may work together on some.
- **Choice:** Let children attempt the problems with which they feel comfortable. Children must feel a sense of confidence and pleasure in attempting such problems. A puzzle or investigation loses its charm when it is much too complex to understand or is forced upon children. However, a teacher can often find various ways of stimulating interest in the problem.
- **Skill Set:** Skills needed to solve these problems are not entirely age dependent nor are they sequential in nature. Diversity in skills and strategies employed should be recognised, appreciated and shared.
- **Representation and communication:** Encourage children to discuss and communicate. Let them ask 'What if' questions. Help them to represent their solutions in the form of drawings and present their solutions to their classmates at the end. Focus needs to be on building reasoning, guessing and testing, explaining and summarising skills.
- **Lead Questions:** I have introduced the problems through a series of questions. Some children may need more questions to understand and experiment with the problem. Some may not need more than one question to begin to explore. The teacher can intervene when the child seems to be stuck.
- **Themes:** I have used five themes. Three of the themes relate to number manipulation skills: Investigations with Hundreds Square, Magic Figures, Missing Digits. Two themes relate to spatial skills: Dot Paper Activities and Toothpick Posers.

Hundred Square Grid Investigations

The focus of the activities is on:

- Noticing relationships between numbers of a small set;
- Experimenting with the numbers in a given set to discover patterns;
- Verifying that the discovered patterns and relationships hold in similar settings;
- Generating new questions by extending the activity to different sized squares;
- Exploring the same activity in new settings;
- Viewing from a different orientation;
- Finding multiple paths in a systematic manner and recording the information.

Please see Figure 1

Figure 1

Look at the circled (diagonal) numbers. 1, 12, 23,

What pattern do you notice?

Look at other diagonal numbers. 11, 22, 33...

21, 32, 43...

Does the pattern repeat for all diagonals?

Please see Figure 2.

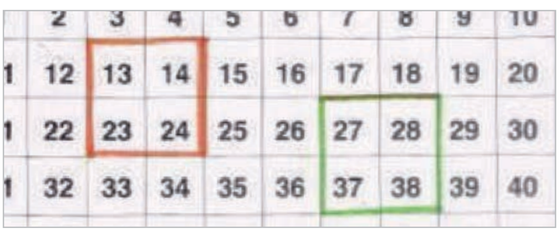

Figure 2

Look at the numbers in any 2 by 2 square as shown in the figure.

Sum the numbers horizontally.

Sum the numbers vertically.

What sums do you get? Now sum the numbers diagonally. What sum do you get?

Now select any other 2 by 2 square from the number grid.

What is the difference between the two horizontal sums? Did you get the same difference as you did the previous time?

Does the same relationship hold for the difference between the two vertical sums?

Can you explain your findings?

Please see Figure 3.

Look at the numbers in any 3 by 3 square as shown in the figure.

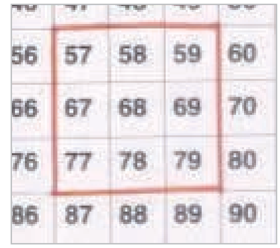

Figure 3

Sum the numbers horizontally. What do these sums add up to?

Sum the numbers vertically. What do these sums add up to?

Will this be so for other 3 by 3 squares? Try and see.

Please see Figure 4.

Sum the diagonal numbers of this figure.

What do these sums add up to?

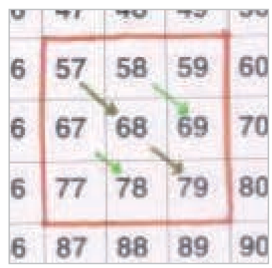

Figure 4

Please see Figure 4a.

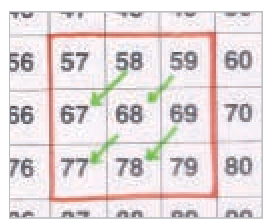

Figure 4a

Sum the diagonal numbers of this figure.

What do these sums add up to?

Will this be so for other 3 by 3 squares? Try and see.

Please see Figure 5.

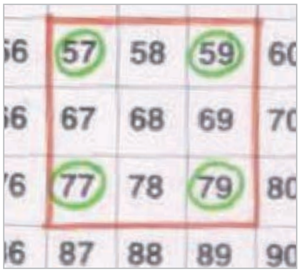

Figure 5

Try summing numbers in the opposite corners (circled ones in pairs). What do you see?

What about the remaining numbers? Which other number pairs have the same sum?

What about the number in the centre? Is there any connection between this number and the sum?

See if the same relationships hold with any other 3 by 3 square on the grid.

Please see Figure 6.

Now let us experiment with multiplication.

Figure 6

Multiply the numbers of the opposite corners. Note the results.

Do the same with another 3 by 3 square. Note the results.

Do you see any pattern in the products?

Please see Figure 7.

Now try multiplying other pairs as shown in the figure. Note the results.

What do you notice about the differences between the products?

Do the same with another 3 by 3 square. Compare the results.

Please see Figure 8.

Figure 7

Select a rectangular shape (3 by 4) and circle the four corner numbers.

What relationships do you see here between the products of pairs of these numbers?

Please see Figure 9.

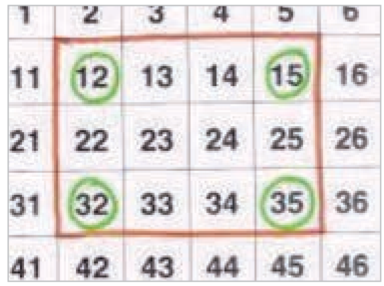

Figure 8

Will these relationships hold in a parallelogram shape?

Please see Figure 10.

Figure 9

Select any 4 by 4 square as shown in the figure. Add the four corner numbers. Write down your total.

What do the four centre numbers add up to?

Can you find another 2 by 2 square in the number grid which adds up to the same total?

Will this work for other 4 by 4 squares?

Will this work for squares of other sizes?

Please see Figure 11.

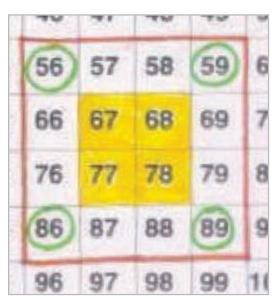

Figure 10

Try to do this in your mind first. Later you can verify your answers by placing one grid over another.

A tracing of a hundred square is rotated a half turn clockwise (i.e., the way a clock's hands move) and placed on the original. The two corresponding numbers in each cell are then added together.

What numbers are produced in the first few rows? The second row? Are they the same?

What if you tried a quarter turn (a 90 degree rotation), clockwise?

100	99	98	97	96	95	94	93	92	91
81	82	83	84	85	86	87	88	89	90
80	79	78	77	76	75	74	73	72	71
61	62	63	64	65	66	67	68	69	70
60	59	58	57	56	55	54	53	52	51
41	42	43	44	45	46	47	48	49	50
40	39	38	37	36	35	34	33	32	31
21	22	23	24	25	26	27	28	29	30
20	19	18	17	16	15	14	13	12	11
1	2	3	4	5	6	7	8	9	10

Figure 11

Please see Figure 12.

Square numbers

Shade the square numbers in a number grid.

Write down the square numbers in order.

Do you see any pattern in the way they are increasing?

Which column has only one square number?

Which columns have two square numbers? Why?

Why are there no square numbers in certain columns? What digits do you see in the units' places of these columns? Will numbers ending with such digits always be non-square numbers?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 12

Knight's Tour of a Number Board

[Explain the knight move if so needed; see https://en.wikipedia.org/wiki/Knight_%28chess%29.]

Please see Figure 13.

Can you go from 1 to 100 in knight moves?

If you add up the numbers you land on as you go, what is the minimum total? The maximum?

The minimum *even* total? The minimum *odd* total?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 13

Dot Paper Explorations

The focus of these activities is on the following:

- Visualising shapes likes squares and triangles in dot array;
- Realising that not all squares will have sides parallel to the base of the paper;
- Sharpening sense of congruence (same shape and size);
- Counting in a systematic manner.

Please see Figure 14.

Look at the 2 by 2 square as shown in the figure in your dot paper. How many squares can be seen in the figure? (Here we mean: squares of all possible sizes.) The answer is 5 (four squares of size 1 by 1, and one square of size 2 by 2).

Figure 14

Draw a 3 by 3 square. How many squares can be seen in it?

Try with a 4 by 4 square. Do you see a pattern in the numbers?

Now try with a 2 by 3 rectangle.

How many squares are there?

Try with a 3 by 4 rectangle.

Please see Figure 15.

In a 5 by 5 dotted region, how many different sized squares are possible?

[Children should be able to find at least 6 of them. There are 8 different sizes.]

Figure 15

Please see Figure 16.

This square has 8 dots on its outline and 1 dot inside the square.

Can you make a square with 12 dots on the outline and 4 dots inside?

Can you make a square with 4 dots on the outline and 1 dot inside?

Can you make a square with only 2 dots inside? 5 dots?

Can you make a triangle with one dot inside?

How many different triangles can be drawn which have just one dot inside?

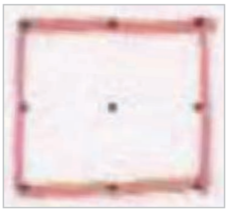

Figure 16

Please see Figure 17.

Can you make a figure with the same area but greater perimeter?

Can you make a figure with the same area but smaller perimeter?

Can you make a figure with the same perimeter but greater area?

Can you make a figure with the same perimeter but smaller area?

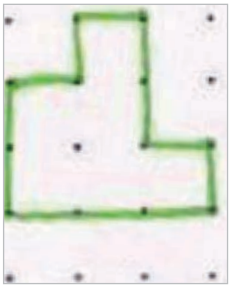

Figure 17

Please see Figure 18.

Look at the spiral made by an ant. It takes a step of 1 unit length, takes a 90 degree turn to the right, takes a step of 2 units length, takes a 90 degree turn to the right, takes a step of 3 units length, takes a 90 degree turn to the right, and then repeats a step of 1 unit length ....

The 1, 2, 3 ant reaches back to the starting point.

Will a 1, 3, 2 ant reach its starting point too?

How about a 3, 1, 2 ant?

Now create your own ant and see if it reaches back to its starting point.

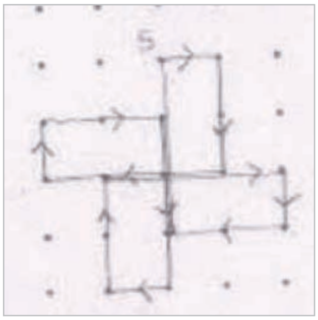

Figure 18

What would the figure look like if the ant uses just 1 number from start to end?

What would the figure look like if the ant uses just 2 numbers from start to end?

What would the figure look like if the ant uses 4 numbers?

Please see Figure 19.

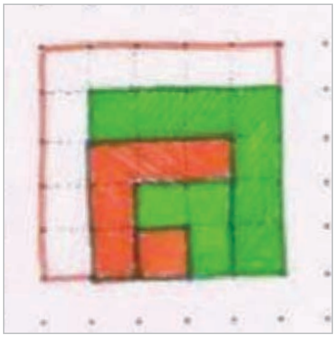

Figure 19

Look at the way the figure is growing.

Copy it into your dot paper and make the 5th pattern and the 6th pattern.

How many squares are added to form the 5th pattern?

How many squares are added to form the 6th pattern?

How many unit squares are there in the first stage?

How many unit squares are there altogether in the second stage?

How many unit squares are there altogether in the third stage?

How many unit squares are there altogether in the fourth stage?

How many unit squares are there altogether in the fifth stage?

Can you see a pattern?

Please see Figure 20.

In how many different ways can I put 5 squares together? One way is shown in the figure. Two squares should share a common edge. Such shapes are called ‘Pentominoes’.

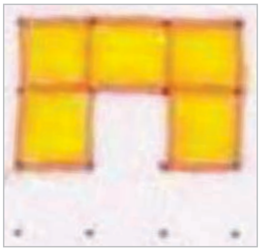

Figure 20

Please see Figure 21.

Look at the pictures.

How many different ways can the ant return to its nest?

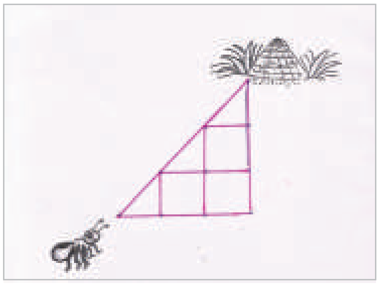

Figure 21

Please see Figure 21a.

How many different ways can the dog return to its kennel?

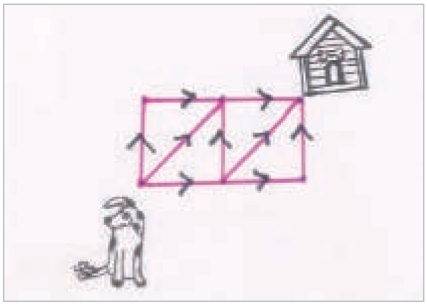

Figure 21a

Please see Figure 21b.

The bee starts at one corner and tries to pass through as many points (flowers) as possible before it reaches the hive.

How many flowers can it visit?

You can make up more such questions and investigate.

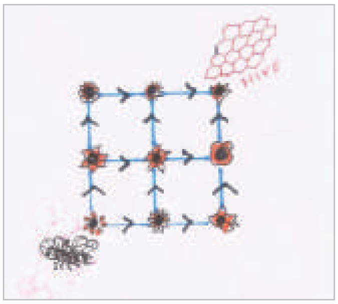

Figure 21b

Magic Sums

Please see Figure 22.

Materials: Cards with circles set out in a V-shape (5 circles card, 7 circles card), Counters (numbered 1 to 10) to fit into the circles

These problems are easier to work with when there are numbered counters. Children should not be required to copy drawings or write and erase numbers while working on them. However they can record their solutions on paper.

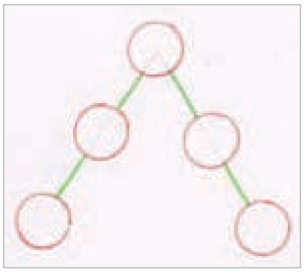

Figure 22

Place each of the numbers 1 to 5 in the V-shaped card so that the two arms of the V have the same total.

How many different ways can you do it?

Is there anything common to all your solutions? Can you explain why? What can you say about the number pairs that appear on the arms?

Now place the numbers 2 to 6 in the V-shaped card so that the two arms of the V have the same total.

How many different ways can you do it?

Is that the same as in the earlier case?

Are the relationships in these solutions similar to the earlier solutions? Now try with other combinations of 5 consecutive numbers.

Can you quickly figure out the number that should go into the bottom circle where both the arms meet?

Try the same with any 5 consecutive even numbers or 5 consecutive odd numbers.

Here is a V card with arms of length 4.

Please see Figure 23.

Place each of the numbers 1 to 7 in the **V-shaped** card so that the two arms of the V have the same total.

Try again with a set of seven consecutive numbers starting with an even number (4, 5, ...).

Now try with a set of consecutive numbers starting with an odd number. (7, 8, 9,...).

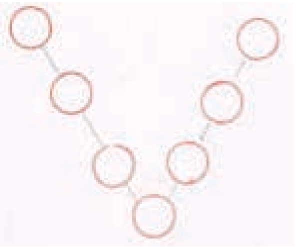

Figure 23

Now let us try a similar exercise in a new design. You can record the results in a square paper.

Please see Figure 24.

Place each of the numbers 1 to 5 in the **Plus**-shaped card (with 5 squares) so that the row and the column have the same total.

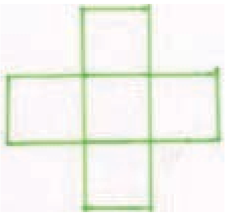

Figure 24

Please see Figure 25.

Place the numbers 1 to 9 in the **Plus**-shaped card (with 9 squares) so that each of the four arms of the plus has the same total. How many different solutions are possible?

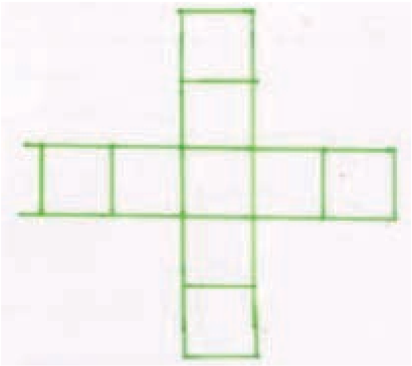

Figure 25

Please see Figure 26.

How will you arrange the numbers 1 to 7 so that the three arms have the same total?

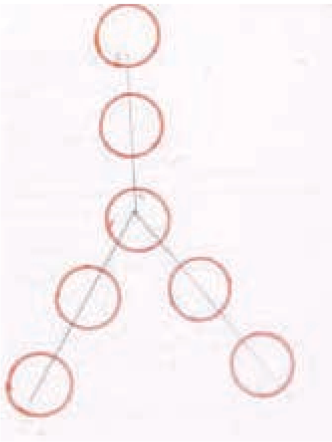

Figure 26

Please see Figure 27.

Arrange the numbers 1 to 6 in each of the arms.

The sum of each side of the triangle should equal 9.

Can you arrange them so that all the sides total 10? 11?

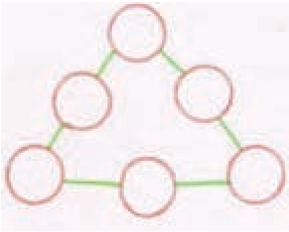

Figure 27

Please see Figure 28.

Materials: Circle cards with three rings.

Can you place three different numbers in the rings so that the difference between each pair is odd?

Can you place three different numbers in the rings so that the difference between each pair is even? What do you notice about the sum of each pair in each case?

Try with a circle card with 4 rings.

Is it possible to place 4 different numbers in the rings so that the differences between neighbouring pairs of numbers are all odd?

Is it possible to place 4 different numbers in the rings so that the differences between neighbouring pairs of numbers are all even?

Can you say why?

Now try a circle card with 5 rings. What do you notice?

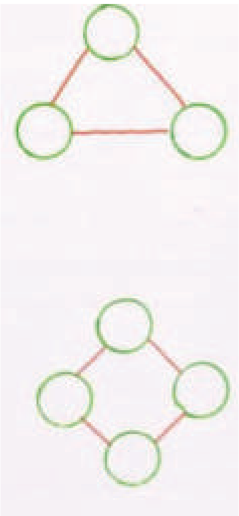

Figure 28

Toothpick Problems

Materials: Toothpicks of the same size.

Skill set: Develops reasoning and exercises spatial skills.

Building with toothpicks:

How many right angles can you make using 2 toothpicks?

Can you cross 2 toothpicks to create 3 different angles?

Take 6 toothpicks. Can you make a star with them?

Removing or moving toothpicks:

Please see Figure 29.

Can you remove 2 toothpicks to leave only 2 squares?

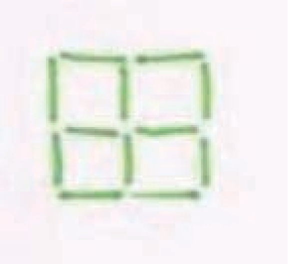

Figure 29

Please see Figure 30.

Can you move 4 toothpicks to make 6 triangles?

Correcting a statement:

Can you move a single toothpick in each case to correct the following statements?

XI – V = IV, X + V = IV, XIV – V = XX, L + L = L

Figure 30

Word problems and toothpicks:

I used 50 toothpicks to make some squares and triangles. No two of the shapes touched on another.

I made 15 shapes in all.

How many squares did I make?

Please see Figure 31.

A farmer used 13 toothpicks to make a model of 6 identical sheep pens. Unfortunately, one of the toothpicks was broken. Use 12 toothpicks to show how the farmer can still make 6 identical pens.

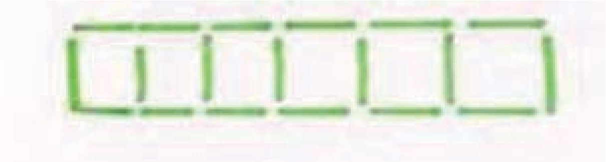

Figure 31

Patterns with toothpicks

Please see Figure 32.

Look at the squares made of tooth picks.

How many toothpicks do you need to make a 1 by 1 square?

How many toothpicks do you need to make a 2 by 2 square?

How many toothpicks do you need to make a 3 by 3 square?

How many toothpicks do you need to make a 4 by 4 square?

Do you see a pattern in the numbers?

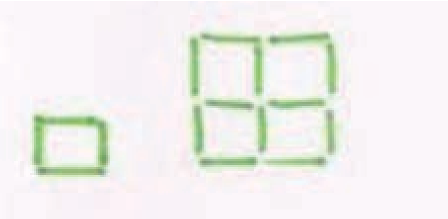

Figure 32

Missing Digits

Please see Figures 33, 34, 35, 36, 37, 38.

What numbers will go into the empty spaces?

Figure 33

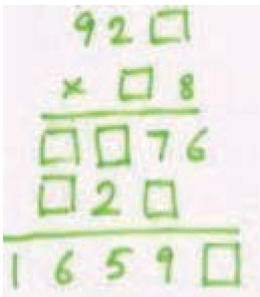

Figure 34

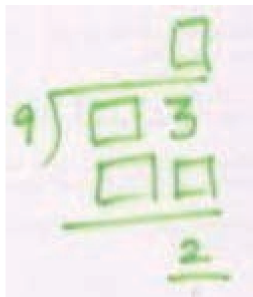

Figure 35

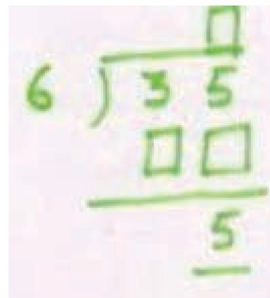

Figure 36

Figure 37

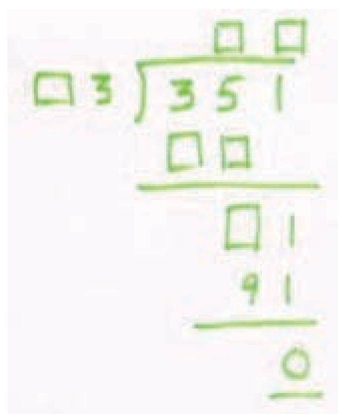

Figure 38

Sources

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- An ATM Activity Book – Association of Teachers of Mathematics
- Teachers Resource Information Pack – Hampshire

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at padmapriya.shirali@gmail.com

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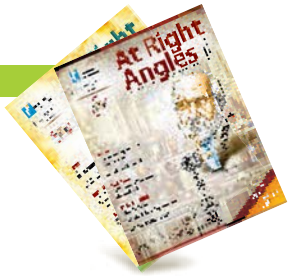

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