Aryabhata and the Construction of the First Trigonometric Table

VIJAY A. SINGH &
ANEESH KUMAR

Introduction

We routinely use our calculator to obtain the sine of a given angle. (In earlier times we used published tables!) Little do we realize that a debt of gratitude is owed to the fifth century Indian savant Aryabhata who was the first to tell us about the sine function and create the first table of sines — in his seminal work, the *Aryabhatiya* (499 CE). In this article, we describe the trigonometric identities used by Aryabhata to obtain the table of sines.

The Indian mathematical tradition is largely word-based. Results are mentioned but derivations are omitted. The *Aryabhatiya* with more than 100 cryptic, super-compressed verses of dense mathematics is a prime example. In order to make our presentation pedagogical we take a unit circle and measure angles in degrees or radians instead of the (now) archaic notation in the *Aryabhatiya* and its commentators. We show that Aryabhata's sine table entails taking the difference of the sine of two closely spaced angles and then taking the second sine difference. We also show that the trigonometric identities are the same as those in the finite difference calculus one uses nowadays to numerically obtain the first and second derivatives of the sine function. We follow this with a brief discussion. An understanding of these identities and preparation of the sine table will enable the reader to get an appreciation of the path-breaking work of Aryabhata. At the end, we suggest a few problems and invite the reader to try their hand at solving them.

Keywords: Aryabhata, Sine, Sine Table, Finite Difference Calculus

The *Aryabhatiya* consists of 121 cryptic verses, dense and laden with meaning [1, 2]. The work is divided into 4 parts or *padas*: the *Gitikapada* (13 verses), *Ganitapada* (33 verses; the mathematics part), the *Kalkriyapada* (25 verses) and *Golapada* (50 verses; the astronomy part, which is better known than the others). There are two verses in *Ganitapada* describing the solution of the linear Diophantine equation. This has received due recognition. Our focus will be on the trigonometry part in *Ganitapada* which in our view has suffered neglect.

Virtually every major Indian mathematician has commented on the *Aryabhatiya*. Often it is in terms of a formal *Bhashya* (Commentary). Table I lists some of them. Notable among them is the voluminous work *Maha Bhashya* of the 15th century mathematician Nilakantha Somaiyaji (1444–1544 CE). He was part of the Kerala school which, beginning with Madhava (1350–1420 CE), founded the calculus of trigonometric functions.

Bhaskara I	629 CE	Sanskrit	Valabhi, Gujarat
Suryadeva Yajvan	1191 CE	Sanskrit	Gangaikonda-Colapuram
Parameshvara	1400 CE	Sanskrit	Allathiyur, Kerala
Nilakantha Somayaji	1500 CE	Sanskrit	Trikandiyur, Kerala
Kondadarma	unknown	Telugu	Andhra
Abul Hasan Ahwazi	800 CE	Arabic	Ahwaz, Iran

Table 1. A host of eminent mathematicians have commented on the *Aryabhatiya* written 499 CE. Some, like Brahmagupta (600 CE) or Bhaskara II (1100 CE), have not written a specific commentary but have dwelt extensively on it. The above is an abbreviated list of specific commentaries and the dates are approximate. Our main source is the work of K. S. Shukla and K. V. Sarma [2] which cites around 20 commentaries.

The *Ardha-Jya* or Sine Function

Aryabhata lays down — for the first time in the history of mathematics — a definition of the sine function. He poetically describes the sine function as the half bow-string or the *Ardha-Jya*, and relates the cosine functions to the arrow or *saar* (see Figure 1). This is not the only example of poetry making an appearance in his mathematics. To describe the fact — heretical and revolutionary for those times and for long afterwards — that the Earth is rotating and the Sun is stationary, Aryabhata evokes the tranquil metaphor of a boat floating down the river and the stationary river bank which seems to move backwards. Also being a poet, Aryabhata composed the *Aryabhatiya* in verse form with over 100 verses, respecting the norms of grammar and metre.

The sine function is the half-chord AP of the unit circle in Figure 1:

$$\sin\theta = \frac{AP}{OA} = AP \qquad (OA = 1).$$

The circle may be large or small; correspondingly, AP and OA may be large or small, but the LHS is a function of θ and is not dependent on the scale of the figure. All metrical properties related to the circle can be derived using trigonometric functions and the Pythagorean theorem (also known as the ‘Baudhayana’ or ‘Diagonal’ theorem [4]). For example, the geometric properties of a triangle can be related to the arcs of the circumscribing circle using the sine and cosine functions. Or the diagonals of the inscribed quadrilateral can be related to its sides. (A recent proof of the Pythagorean theorem using the the law of sines suggests that all metrical properties of a circle can by obtained using trigonometry alone [5].) By emphasizing the role of the half-chord, Aryabhata endowed circle geometry with metrical properties. This alone should qualify him as the founder of trigonometry. But he did more.

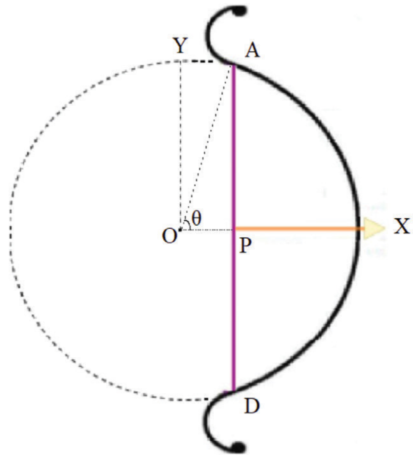

Figure 1. The bow superposed on the unit circle. Half the bow string or half chord AP is $\sin\theta$ as defined by Aryabhata. OP is $\cos\theta$, while $PX = 1 - \cos\theta$ is called the *saar*. See text for comments.

Note that the length of the half-chord AP is close to the length of the arc AX when the angle is small (i.e., $\sin\theta \approx \theta$ if θ is small and in radians). This was known to Aryabhata. Similarly, $\sin 90^\circ = 1$, since then the half-chord is the same as the radius. In the 9th verse of *Ganitapada* he uses the property of an equilateral triangle and obtains $\sin 30^\circ = 1/2$.

We pause to note that Aryabhata also states the value of π as $62832/20000$ in the 10th verse. This value is 3.1416 and he is careful to state that this is 'proximate' (*Asanna*) suggesting that we can obtain better values for π with more effort. Here, the word *Asanna* or 'proximate' is to be distinguished from *Sthula* which is approximate or roughly equal.

The Difference Formula for Sine and Cosine

The 12th verse of *Ganitapada* plays a central role in the tabulation of the sine function. It is cryptic and to unravel its meaning we first need to obtain the difference formula for the sine. The presentation below relies on a number of sources: (i) The commentary of Nilakantha Somaiyaji [3]; (ii) the treatment of Shukla and Sarma [2]; (iii) and for the sake of ease of understanding we follow Divakaran [4] and work with a unit circle rather than one with radius 3438[1].

Figure 2 depicts a quadrant of the unit circle where $OX = OY = 1$. The arcs XA, XB and XC trace angles θ, $\theta + \phi$ and $\theta - \phi$ respectively. The half-chords AP, BQ and CR are the corresponding sine functions. We drop a perpendicular CS from the circumference to the half-chord BQ as shown. According to his commentator Nilakantha Somaiyaji [3], Aryabhata obtained the relationship between the difference in the trigonometric functions by demonstrating that $\triangle BSC$ and $\triangle OPA$ are similar and exploiting this property in an ingenious manner. We trace his line of reasoning in the Appendix where we derive

$$\sin(\theta + \phi) - \sin(\theta - \phi) = 2\sin(\phi)\cos\theta. \tag{1}$$

The difference in the sines is thus proportional to the cosine of the mean angle.

$$\cos(\theta + \phi) - \cos(\theta - \phi) = -2\sin(\phi)\sin\theta \tag{2}$$

The difference in the cosines is thus proportional to the (negative) of the sine of the mean angle.

1 A convention adopted by earlier workers. Note that one radian is 3438 minutes; 2π radians is 180°; and 1° is 60 minutes.

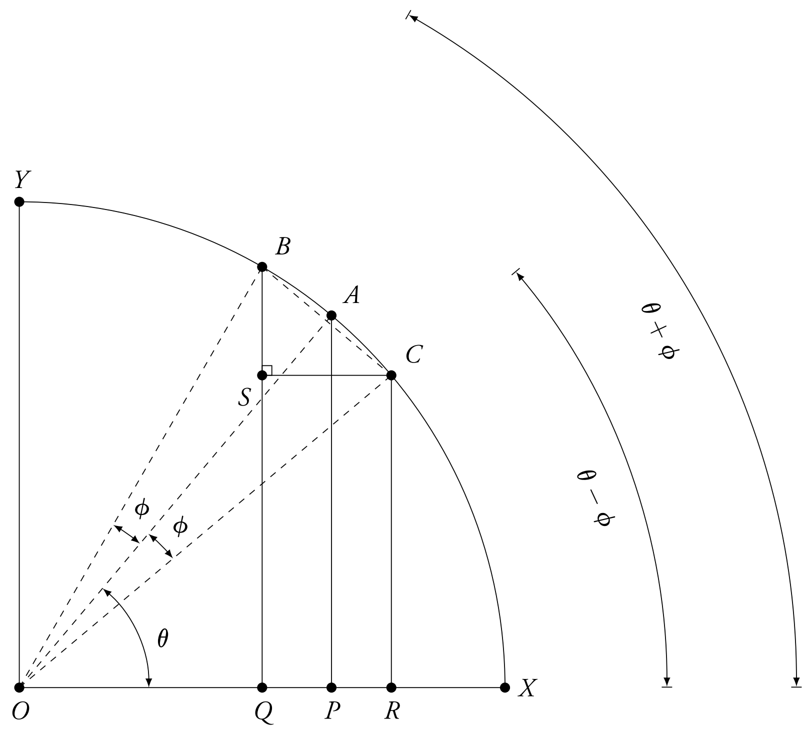

Figure 2. Derivation of the sine difference relation. The figure depicts the quadrant of a unit circle of radii $OX = OY = 1$. The half-chords AP, BQ and CR are $\sin\theta$, $\sin(\theta+\phi)$ and $\sin(\theta-\phi)$ respectively. It is worth noting that (later) we shall take ϕ to be a small angle.

Expressing the difference relations in another way, we get

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right), \text{ and} \tag{3}$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right) \tag{4}$$

The Sine Table

Aryabhata begins with dividing the quadrant of the unit circle into 24 equal parts. He then obtains the values of the sines at fixed angles between 0 and $\pi/2$ thus generating the sine table for $\pi/48 = 3.75°$, $2\pi/48 = 7.5°$, $3\pi/48 = 11.25°$, …up to 90° (See Table 2). This table is given in verse 12 of the *Gitikapada*. It has been used by Indian astronomers (and astrologers!) in some form or another from 499 CE to the present. We shall see now how the table was generated.

As the quadrant is divided into 24 equal parts, let us take $\varepsilon = \frac{\pi}{48}$. Let us take $\phi = \varepsilon/2$ where ε is small. We now compute the differences of successive values of sine and cosine for the increment of ε. In other words, we compute the differences δs_n and δc_n defined below.

$$\delta s_n := \sin n\varepsilon - \sin(n-1)\varepsilon, \text{ and}$$
$$\delta c_n := \cos n\varepsilon - \cos(n-1)\varepsilon$$

where n varies from 0 to 24.

Using the equations (3) and (4), we get

$$\begin{aligned} \delta s_n &= \sin n\varepsilon - \sin(n-1)\varepsilon \\ &= 2\sin\left(\frac{n\varepsilon-(n-1)\varepsilon}{2}\right)\cos\left(\frac{n\varepsilon+(n-1)\varepsilon}{2}\right) \\ &= 2\sin\left(\frac{\varepsilon}{2}\right)\cos\left(\left(n-\frac{1}{2}\right)\varepsilon\right) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \delta c_n &= \cos n\varepsilon - \cos(n-1)\varepsilon \\ &= -2\sin\left(\frac{n\varepsilon-(n-1)\varepsilon}{2}\right)\sin\left(\frac{n\varepsilon+(n-1)\varepsilon}{2}\right) \\ &= -2\sin\left(\frac{\varepsilon}{2}\right)\sin\left(\left(n-\frac{1}{2}\right)\varepsilon\right) \end{aligned} \tag{6}$$

We introduce the shorthand s_α to write $\sin(\alpha\varepsilon)$ and c_α to write $\cos(\alpha\varepsilon)$ and restate the equations (5) and (6) as

$$\delta s_n = s_n - s_{n-1} = 2s_{\frac{1}{2}}c_{(n-\frac{1}{2})} \tag{7}$$

$$\delta c_n = c_n - c_{n-1} = -2s_{\frac{1}{2}}s_{(n-\frac{1}{2})} \tag{8}$$

The above is a pair of coupled difference equations, and it was Aryabhata's insight to take the second difference, namely

$$\begin{aligned} \delta^2 s_n &= \delta s_n - \delta s_{n-1} = 2s_{\frac{1}{2}}(c_{(n-\frac{1}{2})} - c_{(n-\frac{3}{2})}) \\ &= -4s_{\frac{1}{2}}^2 s_{n-1} \qquad \text{from equation (8)} \end{aligned} \tag{9}$$

Thus the second difference of the sines is proportional to the sine itself. The next step is to represent the RHS in terms of a recursion. We observe that s_n on the RHS of equation (9) may be written as

$$\begin{aligned} s_n &= s_n + (s_{n-1} - s_{n-1}) + (s_{n-2} - s_{n-2}) + \cdots + (s_1 - s_1) + (s_0 - s_0) \\ &= (s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_1 - s_0) + s_0 \\ &= \delta s_n + \delta s_{n-1} + \cdots + \delta s_1 + 0 \\ &= \sum_{m=1}^{n} \delta s_m \end{aligned}$$

Thus

$$\delta^2 s_n = \delta s_n - \delta s_{n-1} = -4s_{1/2}^2 \sum_{m=1}^{n-1} \delta s_m \tag{10}$$

Thus we get a recursion relation where the second difference of the sines is expressed in terms of all previously obtained first and second sine differences. To initiate the recursion, we need δs_1 which is $s_1 - s_0 = \sin\varepsilon - \sin 0 \approx \varepsilon$, since for small angles the half-chord and the corresponding arc are equal, as stated in the previous section.

Using the recursion relation we can generate Aryabhata's celebrated sine table, taking $\pi = 3.1416$ and $\sin\varepsilon = \varepsilon = 0.0654$ $(= 225')$.

Table 2 depicts some typical values of the sine function as well as the value of the sine multiplied by 3438 (the so called '*Rsine*' of Aryabhata). We see that this matches Aryabhata's sine table to ± 1 minute. For example, $\theta = \pi/6$ gives 1719 minutes. For comparison, we also give the current accepted value of $\sin\theta$ to four decimal points. Note that Aryabhata takes angles only till $\pi/2$; he seems aware of the fact that going further is unnecessary given the periodic nature of the sine function.

θ	$\sin\theta$ (Aryabhata)	$\sin\theta$ (minutes)	$\sin\theta$ (modern)
$\pi/48$	0.0654	225	0.0654
$2\pi/48$	0.1305	449	0.1305
$3\pi/48$	0.1951	671	0.1951
$4\pi/48$	0.2588	890	0.2588
$5\pi/48$	0.3214	1105	0.3214
$6\pi/48$	0.3827	1315	0.3827
$7\pi/48$	0.4423	1520	0.4423
$8\pi/48$	0.5000	1719	0.5000
$9\pi/48$	0.5556	1910	0.5556
$10\pi/48$	0.6088	2093	0.6088
$11\pi/48$	0.6594	2267	0.6593
$12\pi/48$	0.7072	2431	0.7071
$13\pi/48$	0.7519	2585	0.7518
$14\pi/48$	0.7935	2728	0.7934
$15\pi/48$	0.8316	2859	0.8315
$16\pi/48$	0.8662	2978	0.8660
$17\pi/48$	0.8971	3084	0.8969
$18\pi/48$	0.9241	3177	0.9239
$19\pi/48$	0.9472	3256	0.9469
$20\pi/48$	0.9662	3322	0.9659
$21\pi/48$	0.9812	3373	0.9808
$22\pi/48$	0.9919	3410	0.9914
$23\pi/48$	0.9983	3432	0.9979
$24\pi/48$	1.0005	3439	1.0000

Table 2. Table of sine values using Aryabhata's method, $\varepsilon = \pi/48 = 3.75° = 225'$ and $\pi = 3.1416$ and comparison with modern day values. In column 3 we quote values in minutes as done in Verse 12 of the *Gitika* chapter of *Aryabhatiya* [1, 2].

Finite Difference Calculus

Of greater relevance is the fact that the sine (or cosine) difference formulae foreshadow finite difference calculus, a popular numerical technique in this age of computation. Rewriting equations (1) and (2) with $\phi = \varepsilon$,

$$\frac{\sin(\theta+\varepsilon)-\sin(\theta-\varepsilon)}{2\sin(\varepsilon)} = \cos\theta \tag{11}$$

$$\frac{\cos(\theta+\varepsilon)-\cos(\theta-\varepsilon)}{2\sin(\varepsilon)} = -\sin\theta \tag{12}$$

Aryabhata took ε to be $\pi/48$. But he also stated that its value is *yateshtani* or 'as per our wish' (Verse 11, *Ganitapada*). Some took it to be $\pi/96$; others (like Brahmagupta) took it as $\pi/12$ or 15°. If we take ε to be sufficiently small, we have our classic formula for finite difference calculus. Noting that $2\sin\frac{\varepsilon}{2} \approx \varepsilon$ we have the finite difference version of the derivative of sine,

$$\frac{\sin\theta}{d\theta} = \cos\theta,$$

and similarly for the cosine,

$$\frac{\cos\theta}{d\theta} = -\sin\theta.$$

Let us illustrate this with an example. We know that $\sin 37^\circ \approx 0.6$, and $\sin 30^\circ = 0.5$. The difference in angle is 7° which in radians is 0.122. Thus the derivative of sine of the mean angle 33.5° from equation (11) is

$$\delta\sin\theta/\delta\theta = (0.6-0.5)/0.122 = .82.$$

Looking up the sine table or the calculator yields $\cos 33.5^\circ = 0.83$. Similarly equations (9) yields the second derivatives namely

$$\delta^2\sin\theta/\delta^2\theta \approx -\sin\theta$$

$$\delta^2\cos\theta/\delta^2\theta \approx -\cos\theta$$

The above are now called central difference approximations to the first derivative and the second derivative. Naturally, Aryabhata does not use the term 'finite difference calculus' (or 'calculus'). But similar methods are now used to numerically solve differential equations. The student will recognize the above as a standard solution of the classical simple harmonic oscillator. We note in passing that Newton's II Law and the famous Schrödinger equation of quantum mechanics are both second-order differential equations.

Discussion

One can discern a continuity in Indian mathematics, however tenuous, from pre-Vedic times (prior to 1000 BCE) till the 1800s. A striking example is the influence of *Aryabhatiya* on major Indian mathematicians who followed him including Madhava (1350 CE) who founded Calculus [4]; as also the influence on Aryabhata of the mathematics which preceded him [6, 7].

To reiterate, Aryabhata seems aware that (i) $\sin 0^\circ = 0$; (ii) the sine of a small angle is itself, as the small arc is 'almost equal' to the half-chord; (iii) $\sin 30^\circ = 1/2$ (*Ganitapada* verse 9); (iv) $\sin 90^\circ = 1$; (v) the sine function is periodic, so he prudently does not extend the computation to angles greater than 90°. Then, in a remarkably insightful way, he lays down the recursion relation for sine differences which enables one to generate the sine table. It is this work, more than his solution to the linear Diophantine equation (verses 31 and 32 of *Ganitapada*) which establishes him as a genius and one of the brightest stars in the firmament of world mathematics.

The sine table can also be generated using the half-angle formula. This was demonstrated in the *Panchasiddhantika*, a text written barely 50 years after the appearance of *Aryabhatiya* [8]. As pointed out, a feature of the Aryabhata's difference relation is how modern it is. It can readily be seen to be essentially the same as finite difference calculus. It led to the development of the calculus of trigonometric functions by Madhava (1350 CE) and his disciples along the banks of the Nila river in Kerala. This school is variously called the Nila [4] and the Aryabhata school [7]. Another aspect to note is that Bhaskara II (1100 CE) used the division of the great circle into parts of magnitude $2\pi/96$ to carry out discrete integration and obtain the (correct) expressions for the surface area and the volume of a sphere. Jyesthdeva of the Nila (or Aryabhata) school in his work *Yuktibhasa* derived the same results using calculus (circa 1500 CE). Aryabhata can thus legitimately be called the founder of trigonometry.

To sum up, the *Aryabhatiya* exercised a tremendous influence over Indian mathematicians for over a thousand years. For a book with just over one hundred pithy verses, its legacy remains unparalleled in the scientific world. We hope that our article will give our young audience an introduction to his work and will serve as an inspiration.

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References

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PROF. VIJAY A. SINGH has been faculty at IIT Kanpur (1984-2014) and HBCSE, Tata Institute for Fundamental Research (2005-2015) where he was the National Coordinator of both the Science Olympiads and the National Initiative on Undergraduate Science for a decade. He is a Fellow, National Academy of Sciences, India and was President, the Indian Association of Physics Teachers (2019-21). He is currently a Visiting Professor at CEBS, Mumbai.

ANEESH KUMAR is a Standard XII student at the Dhirubhai Ambani High School, Mumbai, with an interest in Mathematics and Physics.

Appendix: The Derivation of the sine and cosine difference formula

We add Figure 2 once again here (labelled as Figure 3) for the reader's quick reference. As stated in the text the figure depicts a quadrant of the unit circle where $OX = OY = 1$. The arcs XA, XB and XC trace angles θ, $\theta + \phi$ and $\theta - \phi$ respectively. The half-chords AP, BQ and CR are the corresponding sine functions. We drop a perpendicular CS from the circumference onto the half-chord BQ as shown.

We show that $\triangle BSC$ and $\triangle OPA$ are similar. By construction $\measuredangle BSC$ and $\measuredangle OPA$ are each $90°$. Note $OB = OC = 1$ (unit radius) and hence $\triangle OBC$ is isosceles. This implies that

$$\measuredangle OBC = \measuredangle OCB = \frac{180° - 2\phi}{2} = 90° - \phi.$$

Also in $\triangle OBQ$,

$$\measuredangle OBQ = 180° - \measuredangle OQB - \measuredangle BOQ = 180° - 90° - (\phi + \theta) = 90° - \phi - \theta.$$

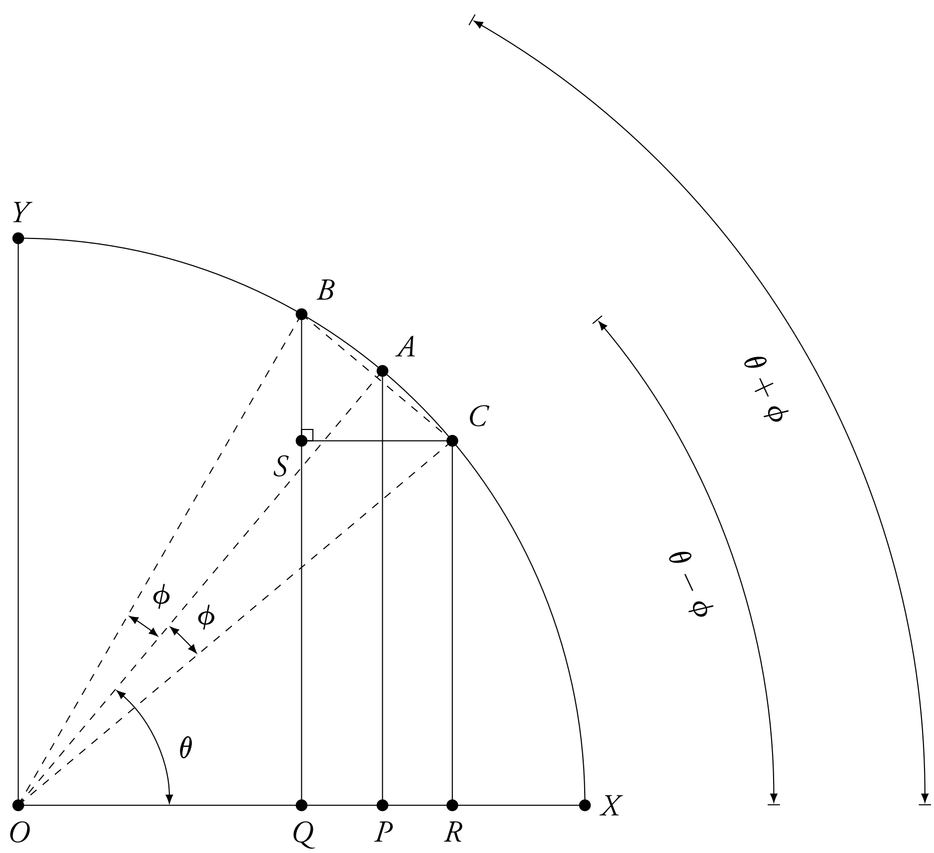

Figure 3. Derivation of the sine difference relation.

Hence $\measuredangle SBC = \measuredangle OBC - \measuredangle OBQ = \theta$. Therefore, $\measuredangle SBC = \measuredangle POA = \theta$. This establishes the similarity of the two triangles by the angle-angle test.

$$\frac{BS}{OP} = \frac{BC}{OA}$$

Equivalently,

$$BS = \frac{BC}{OA} OP \tag{13}$$

On LHS of (13), we have

$$BS = BQ - CR = \sin(\theta + \phi) - \sin(\theta - \phi).$$

On the other hand, on RHS of (13), we have $OA = 1$ (unit radius) and $OP = \cos\theta$. Since $BC \perp OA$, we also have $BC = 2\sin(\phi)$. This yields the sine difference formula (equation (1))

$$\sin(\theta + \phi) - \sin(\theta - \phi) = 2\sin(\phi)\cos\theta.$$

Thus the difference in the sines is proportional to the cosine of the mean angle. We can also obtain the cosine difference formula (equation (1) by noting that

$$\frac{CS}{AP} = \frac{BC}{OA}.$$

Note $AP = \sin\theta$ and $CS = OR - OQ = \cos(\theta - \phi) - \cos(\theta + \phi)$. Hence

$$\cos(\theta + \phi) - \cos(\theta - \phi) = -2\sin(\phi)\sin\theta.$$

The difference in the cosines is proportional to the (negative) of the sine of the mean angle. We pause to note that *prima facie* the two triangles we considered appear unrelated. A hallmark of Indian mathematics is strong geometric intuition and this dates back to the *Sulbasutra circa* 800 BCE. Another is the reliance on the 'rule of three' (*trirasikam*). Here we employ a simple version of it namely, if $a/b = c$ then $a = b \times c$.

Exercises

(1) We can generate the sine table as per Aryabhata's suggestion but not using his value for ε. We choose $\varepsilon = \pi/80 \approx 0.0393$ which is the same as 2.25°. We take $\sin(\varepsilon) \approx \varepsilon$. If you have a calculator, generate all values of sine from 2.25° to 18° in equal steps using equation (10).
   Alternatively, if you have a programmable calculator or a computer, generate all values of sine from 2.25° to 90°. Compare with the results your calculator yields.

(2) In the last section, reference is made of the text *Panchasiddhantika* wherein the half angle formula is mentioned:

$$\cos(2\theta) = 1 - 2\sin^2\theta.$$

How would you (i) derive this by a geometrical construction; (ii) employ this to generate the sine table?