At Right Angles
A Resource for School Mathematics

MASTERY OF MULTIPLICATION

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MASTERY OF MULTIPLICATION

When do we know that a student has mastered a particular concept well? An ability to repeat a learnt process or a practised algorithm is not a necessary indication of mastery. One looks for the ability to apply the concept in situations or contexts, where the concept is embedded indirectly.

Here are a few contexts and problems (not sequenced in any order) which can be presented to students to assess their understanding of the application of the multiplication process. Some involve pure reasoning, while others involve pattern recognition, understanding relationships, multiplication shortcuts, counting, etc.

Problems that involve reasoning require a deep understanding of the concept, in order to handle abstract numbers and operations on them in a flexible manner. These problems are context dependent and cannot be turned into formulae.

Problems that require investigation into the patterns and relationships generated by the multiplication process aid in developing multiplication shortcuts. They help to develop a range of mental strategies. Multiplication shortcuts are used in mental computation which is a part of everyday life.

A problem that involves counting can be presented either through actual models or pictures. There are no standard ways of looking at these models, and they bring forth each student's facility in visualisation.

It is important to keep in mind that the purpose of these activities is not to solve problems but to develop a curiosity and a desire to discover different strategies.

Students may use materials, and drawings to aid in their understanding of the problem situation.

We would recommend that all these problems be solved initially by students working independently. It can then be followed by discussions among the students about the different approaches that they have used to solve them. This provides exposure to the varied ways a problem can be perceived and approached. Let students attempt the strategies tried out by others.

An important learning for the teacher in this process is to become aware of the level of comfort that students have with the types of computations. It also gives scope for teachers to become aware of the thought processes and reasoning employed by the students.

Prerequisite: All these activities presume that students have acquired a basic knowledge of multiples, factors and prime factorisation. Hence, they can be used at the level of class 5 or class 6.

Keywords: Multiplication, Pattern Recognition, Strategizing, Conceptual Understanding

PROBLEM 1

Objective: Logical reasoning

Material: Flashcards

If $6 \times 10 = 60$, what is 12×5?

Students may know the multiplication facts of 12×5 and see that the answer is the same.

Do they see the relationship between the two sets, i.e., 6×10 and 12×5?

Can they explain why the product turns out to be the same?

What is the effect of halving one factor and doubling the other factor?

They could be asked to build more such pairs for verification.

$6 \times 10 = 60$

$12 \times 5 = ?$

Can the students build another pair of factors that relates to the pair 6×10?

30×2 is another such pair. How does it relate to the pair 6×10?

Are the students able to see that 2 is *one-fifth* of 10, and 30 is *five* times 6?

It is good if students notice that the situations are structurally similar. In the first example, the factors got doubled and halved. In the second example, one factor became 5 times while the other became one-fifth.

If $100 \times 9 = 900$, what is 25×36?

How are these two pairs related?

Can the students build other factor pairs that relate to the pair 25×36?

What strategies do the students use to solve the problems?

$100 \times 9 = 900$

$25 \times 36 = ?$

Can the students create more examples to demonstrate this principle?

In this problem and others that follow, one can see the linkage with factors, multiples and prime factorisation.

PROBLEM 2

Objective: Investigation through arrays to discover the doubling and halving strategy of multiplication.

Material: Peg-board or dot-sheet

Here is a visual for how 4 rows and 3 columns are rearranged to form 2 rows and 6 columns.

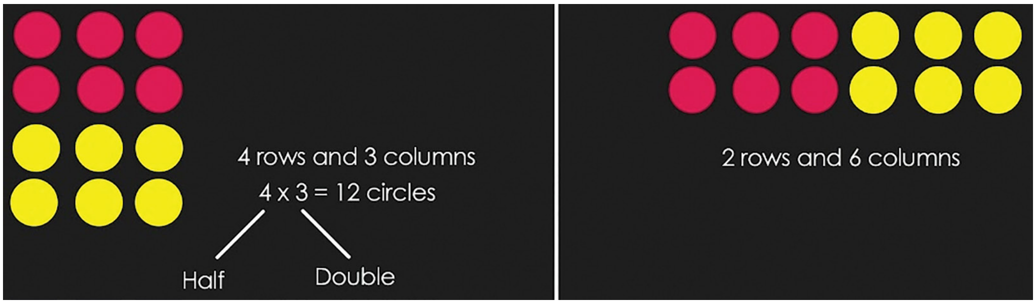

Figure 1

Students can now be asked to build an array to demonstrate what 8 × 6 looks like.

Let them rearrange the pegs as other possible rectangular arrays. How do the other pairs relate to the original pair? i.e., 8 × 6.

2 × 24 (2 is one-fourth of 8 and 24 is 4 times 6)

3 × 16 (3 is half of 6 and 16 is twice 8)

4 × 12 (4 is half of 8 and 12 is twice 6)

What do they observe and conclude?

In the case of both the 3 × 16 and 4 × 12 arrangements, the array has been rearranged by halving one factor and doubling the other.

The halving and doubling strategy involves halving one of the factors and doubling the other.

For example, for 15 × 24, we can double 15 to make 30 and halve 24 to make 12.

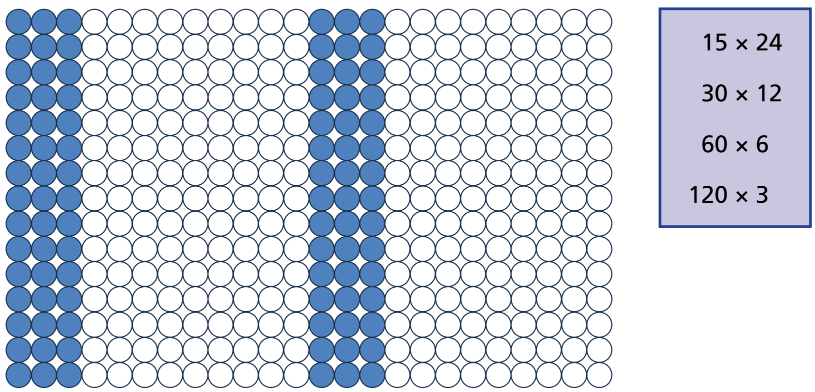

Figure 2

This process can continue till the multiplication becomes easier. 30 can be doubled to make 60 and 12 can be halved to 6.

60 × 6 is easy, that is, 360.

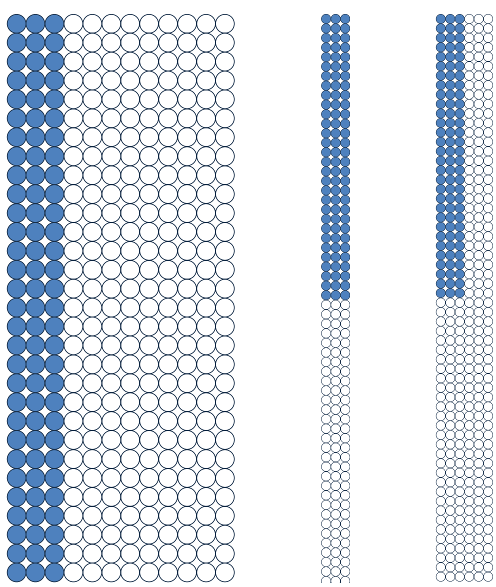

Figure 3

Discuss how this process is helpful in doing multiplication. Let students try the approach for some problems to notice its efficacy.

For what problems does the doubling and halving approach work well?

Let them experiment with more such arrays, e.g., 6 rows, 7 columns, etc. (Where the number of rows is even but the columns are odd.)

Will it work well for 11 × 13? Why would it not work well for this problem?

Will it work if one of the numbers is even?

Let students create some problems where such an approach makes the problem simpler to solve.

Here is one more problem where factorisation by 5 simplifies the problem.

e.g., 375 × 28 = 75 × 140 = 15 × 700, etc.

PROBLEM 3

Objective: Apply understanding of the concept, the laws of associativity, distributivity, etc.

PROBLEM 3.1

Here is a visual demonstration for 8 × 9.

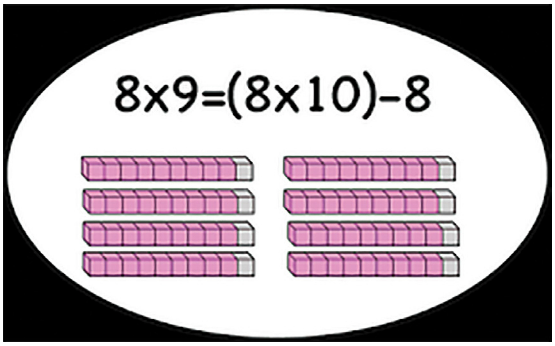

Figure 4

How would the student modify the approach to calculate 18 × 9 or 98 × 9?

PROBLEM 3.2

Are the students able to compute quickly using their understanding of the distributive law? 53 is 3 more than 50. They need to add 9 × 3 i.e., 27 to the given product.

9 × 50 = 450

9 × 53 = ?

9 × 53 = 9 × 50 + 9 × 3

The student's choice of strategy depends on the facility they have with number facts. Strategies are bound to vary.

Problem 3.3

7 × 8 = (5 + 2) × 8,

6 × 7 = (5 + 1) × 7,

9 × 7 = (10 − 1) × 7,

8 × 6 = (10 − 2) × 6

Problem 3.4

Usage of associative property

$8 \times 9 \times 10 \times 11 \times 12$

What strategies will the students use to solve this problem?

Will they regroup the numbers as $8 \times 12 \times 9 \times 11 \times 10$?

$8 \times 12 = 96$ and $9 \times 11 = 99$

The problem has changed to $96 \times 99 \times 10$

Multiplying by 99 can be seen as multiplying by (100 – 1).

$(96 \times 100 - 96 \times 1) \times 10$

$(9600 - 96) \times 10$

$9504 \times 10 = 95040$

Problem 3.5

Associative property is being used here.

$11 \times 12 = 132$

$66 \times 12 = ?$

Problem 3.6

$600 \times 15 = 9000$

$600 \times 45 = ?$

Problem 3.7

How would the students think about approaching these two problems? Discuss the strategies used.

What is 128×8?

What is 26×17?

The strategies for these 2 problems may be different.

A problem such as 128×8 can be attempted in different ways.

$128 \times 8 = 256 \times 4 = 512 \times 2 = 1024 \times 1$

or

$128 \times 8 = 128 \times (10 - 2) = 1280 - 256 = 1024$

Let the students make up similar problems and pose their problems to each other. Encourage them to explain their answers to each other.

PROBLEM 4

Objective: Reasoning out a problem

Pose problems that require reasoning to solve.

Problem 4.1

Two adjacent boxes will represent a double digit number.

Have the students used their understanding of place value?

Do they find more than one solution to this problem?

> Position the digits 3, 4 and 5 to make the product as large as possible:
>
> ☐☐ × ☐ =

Problem 4.2

In what way are these problems alike and in what way are they different?

Fun problem: What is the product of the ten one-digit numbers?

> If $4 \times 6 = 24$ what is $4 \times 600 = ?$
>
> $400 \times 6 = ?$
>
> $40 \times 60 = ?$
>
> $4000 \times 0.6 = ?$

Problem 4.3

Here is another product problem that requires the use of logic in the substitution of letters *a*, *b*, *c*, ... with numbers. Each letter stands for a single-digit number.

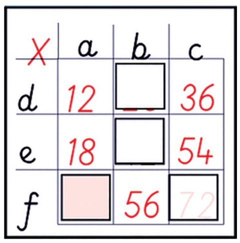

Figure 5

Does this problem have a single solution or more than one solution?

Problem 4.4

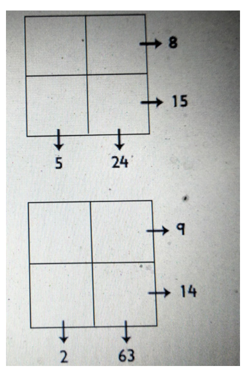

Figure 6

A nice problem where the products are given, and the grid has to be filled with the correct numbers to make the correct products (given on the right and at the bottom).

Problem 4.5

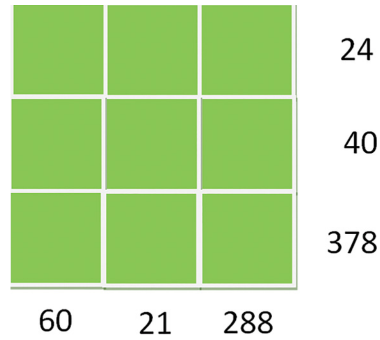

Figure 7

Here is another nice problem from NRICH. (https://nrich.maths.org/11750)

I like the fact that one can deduce the answer with logic and there is hardly any trial and error. It is a good way of reinforcing factors and multiple properties.

Use all the numbers from 1 to 9 in the grid to obtain the given products.

PROBLEM 5

Objective: Discover new relationships.

Let students explore patterns in multiplicative relationships in a series of numbers.

6, 7, 8, 9, 10, 11, 12

What is the relationship of 8 × 10 to 9 × 9? (notice that 8 and 10 are 1 step away from 9)

8 × 10 = 80 which is 1 less than 81.

What is the relationship of 7 × 11 to 9 × 9? (notice that 7 and 11 are 2 steps away from 9)

7 × 11 = 77 which is 4 less than 81.

What is the relationship of 6 × 12 to 9 × 9? (notice that 6 and 12 are 3 steps away from 9)

6 × 12 = 72 which is 9 less than 81.

The discovery of this relationship can be later connected with $a^2 - b^2 = (a + b)(a - b)$

Can the students guess how 5 × 13 relates to 81?

What other observations can they make?

We see that pairs of numbers which are closer together have a greater product.

Now, can students use the fact that 45 × 45 = 2025 to figure out 41 × 49?

Can they explain how this can be done and find the product?

45 × 45 = 2025

41 × 49 = ?

One can build extensions to this discovery by posing further problems.

What is 197 × 197?

Can the students use rounding in this situation? 200 is 3 more than 197. The students can turn the problem into 200 × 194 (shifting by 3 on both sides) which is 38,800. Now they can add

3 × 3 = 9 to the final number to get 38,809.

Let students work with other results and pose problems to one another.

PROBLEM 6

Objective: Multiplication in contexts

Problem 6.1

How many yellow circles?

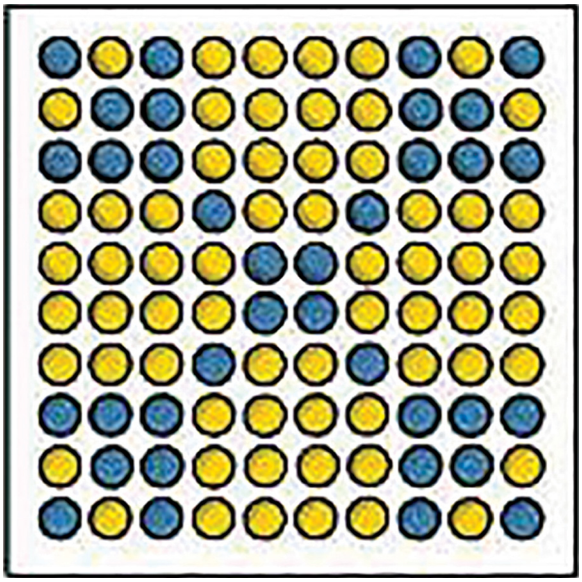

Figure 8

Problem 6.2

How many green squares?

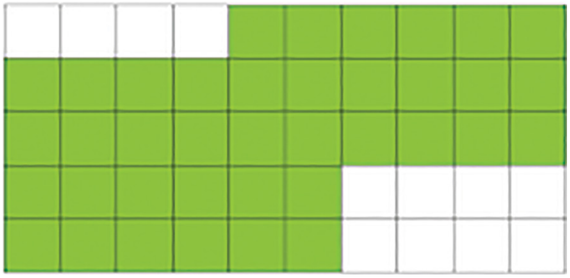

Figure 9

Problem 6.3

How many purple rectangles?

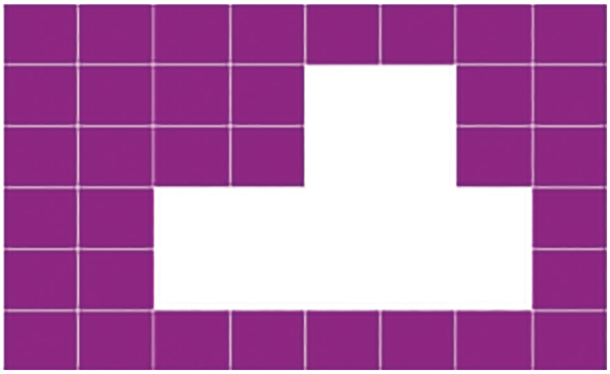

Figure 10

Note the relationship of these problems to area problems.

Problem 6.4

How many seats are on this train?

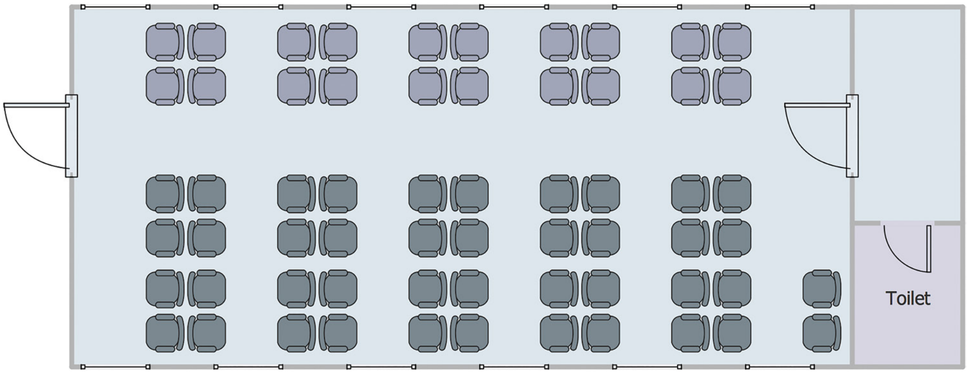

Figure 11

Problem 6.5

How many seats are on this flight?

Figure 12

PROBLEM 7: PEGBOARD ARRANGEMENT

Objective: Using the concept of multiplication in patterns

Here are some pegboard arrangements which children can create and use for counting.

How many pegs have been used in each collection?

Teachers should encourage students to share their different approaches.

Problems can be linked with area.

Here are two patterns where colour may also be used as part of the strategy.

Problem 7.1

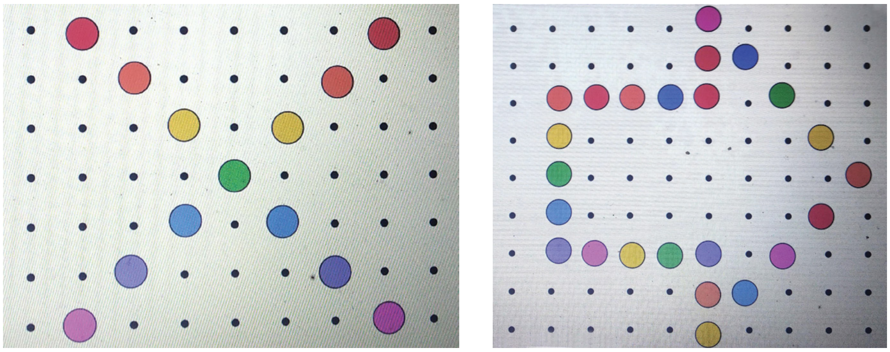

Figure 13

Problem 7.2

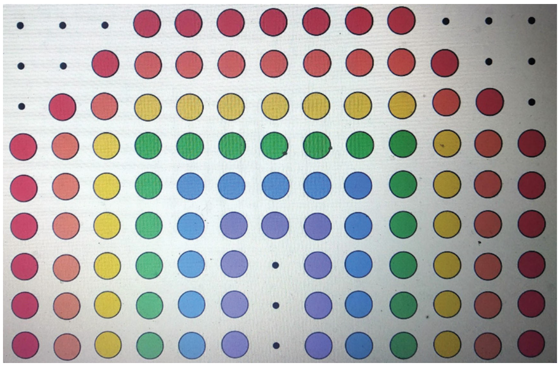

Figure 14

Problem 7.3

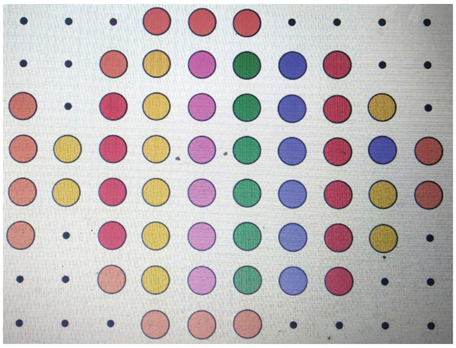

Figure 15

PROBLEM 8: RANGOLI DOTS AND MULTIPLICATIONS

Objective: Using the concept of multiplication in counting

Here is a set of dots made to create a rangoli design.

How many dots has the artist used?

What strategies do the students use for counting?

Let each student figure out a solution and share their strategies.

Would one strategy be to count for one triangle and the central hexagonal shape separately? How would the counting happen for each triangle?

Will the pattern 1, 2, 3, … 7 be considered for summation?

What multiplications are used in the summation of 1 + 2 + 3 + 4 + 5 + 6 + 7?

(1 + 7) + (2 + 6) + (3 + 5) + 4. There are three 8's and one four. 24 + 4 = 28.

There are 6 triangles with 28 dots in each. That is 168 dots in the triangles.

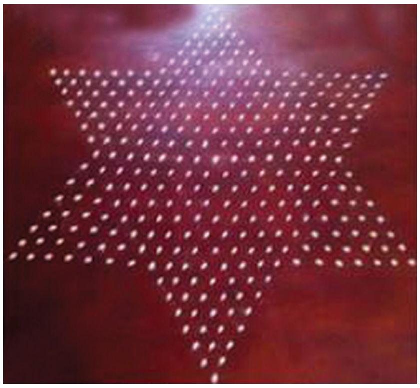

Figure 16

Would the hexagon be counted starting from the diagonal 15, 14, 13, … 8? to get the number of dots for half of the figure.

15 + 14 + 13 + 12 + 11 + 10 + 9 + 8 = (15 + 8) +
(14 + 9) + (13 + 10) + (12 + 11) which is four 23's. That is, 92 dots in half the figure.

The full hexagon has 184 dots.

Altogether this design holds 184 + 168 = 352 dots!

Another strategy may be to use the symmetry of the figure to work out half the dots. The dots are receding from 22, 21, 20, … to 15 with a triangle shape on top.

Are there other ways of counting?

If you had to copy the design, how would you start?

Discuss your strategies and have fun making designs with it!

Here are two more designs for counting.

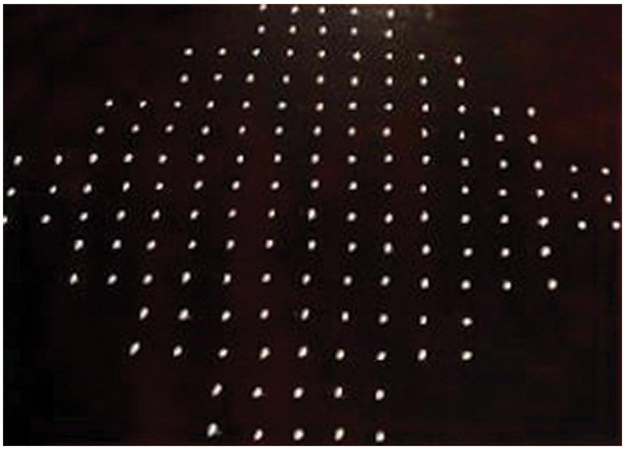

Figure 17

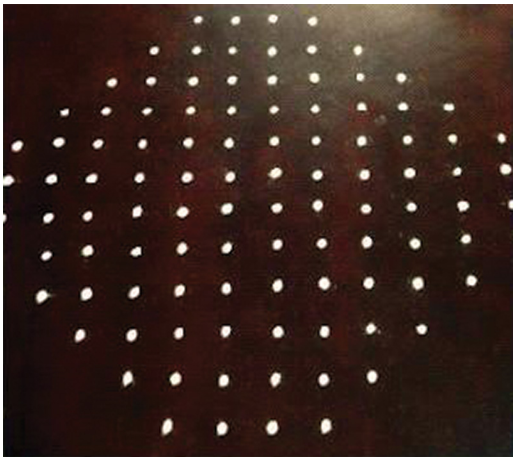

Figure 18

PROBLEM 9: CONSTRUCTIONS WITH CUBES AND MULTIPLICATIONS

Objective: Using the concept of multiplication in counting

Models like these can be made with Jodo cubes or virtually in Mathigon Polypad or https://toytheater.com/cube/

Let students begin with simple cube designs to explain their strategies of counting cubes.

How many cubes?

Most will perhaps count them as 6 + 1 i.e., ($2 \times 3 + 1$).

Problem 9.1

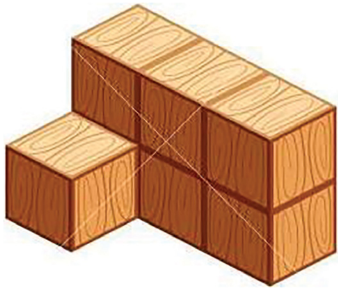

Figure 19

Teachers can connect this problem with volume.

Problem 9.2

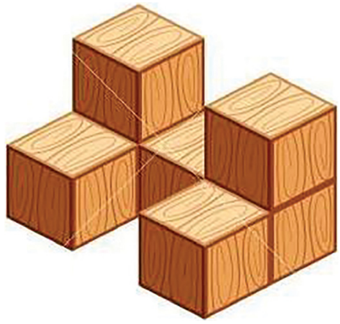

Figure 20

Problem 9.3

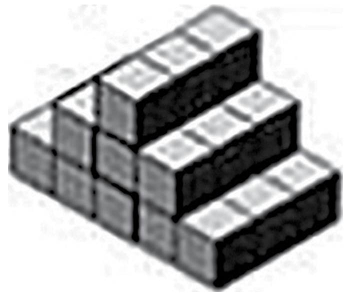

Figure 21

Will this be counted in horizontal layers or vertical slices?

Problem 9.4

How many cubes?

How will they approach this problem?

Is it easier to count the missing ones and subtract from the whole?

Figure 22

Problem 9.5

How many cubes in this E shaped construction?

Figure 23

Did the students resort to counting one by one? Or did they use three rows as 3 fours with 2 extra projections and 2 rows with one cube in each?

Problem 9.6

How many cubes?

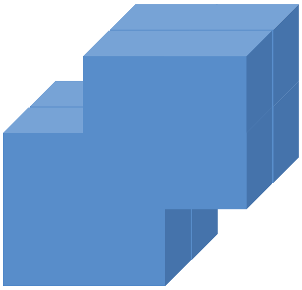

Figure 24

This is an interesting piece for generating discussion.

Some may choose to count them as 2 cubes of size (2×2×2) with an overlap to be deducted. Or will they count in layers?

Problem 9.7

How many cubes?

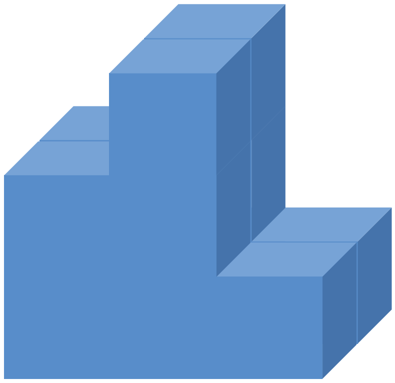

Figure 25

Again, discuss the strategies used.

Here are a few more such examples.

Problem 9.8

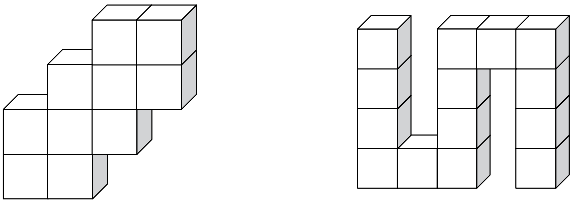

Figure 26

Problem 9.9

What approaches would the students take in counting the cubes in these constructions?

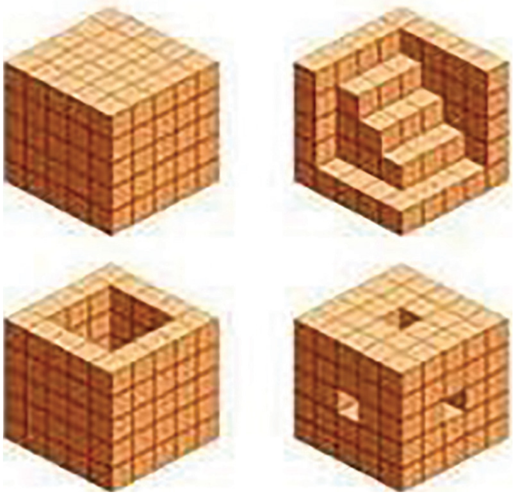

Figure 27

PROBLEM 10

Objective: Prediction of product in multiplication with decimal/fractional numbers

There is often a misconception that many students carry that multiplication always results in a bigger product. Pose problems which require them to estimate and not calculate to check their understanding.

23×0.2	23×2.4	543×0.62
65×0.7	864×1.2	98×0.65

Are the students able to predict where the answers lie?

Will the answer be less than the number or more? Can they give reasons for their thinking?

PROBLEM 11

Objective: Understanding the size of the products

A multiplication grid made to scale may serve as an additional help to students in visualising the numbers and multiplicative relationships.

What patterns do you see in the multiplication table?

Which shapes are squares and which are rectangles?

Does the grid aid to see why 7×9 is one less than 8×8? Or why 4×8 is less than 6×6?

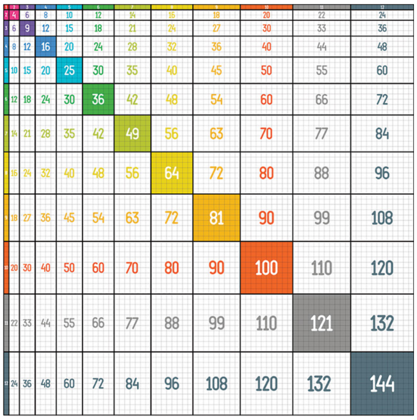

Figure 28

Acknowledgement:

https://www.stem.org.uk/resources/elibrary/resource/32124/multiplication

https://stevewyborney.com

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PADMAPRIYA SHIRALI is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. In the 1990s, she worked closely with the late Shri P K Srinivasan. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box.' She is currently part of the NCERT textbook development group. Padmapriya may be contacted at padmapriya.shirali@gmail.com