Optimizing A Product

A Brilliant Use of Some Lateral Thinking!

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The game "Random Digits" demands Higher Order Thinking Skills (HOTS). The game starts with a common board for all players. The board is essentially for addition, subtraction or multiplication with two multi-digit whole numbers. It specifies the operations and the size, i.e., the number of digits of each whole number used in the operation. However, the exact digits are left blank. As the facilitator names each digit, players have to immediately place them on the board. Once placed, the position of a digit cannot be changed. Zero cannot be a leading digit, i.e., it cannot be placed in the leftmost box of any number. If a player is forced to do that, then s/he is disqualified. If the players choose to maximize the sum, difference or product, then the winner is the one with the maximum result (sum, difference or product). However, the players can choose to aim for the minimum as well. In both cases, players have to think where to place each digit in order to optimize his/her result. It is always a good idea to discuss the optimum result after each game. More details on the game can be found at https://shorturl.at/hkxV3

We were playing this game for a 2-digit × 2-digit multiplication and wanted to maximize the product. The digits given were 2, 5, 8 and 9, not necessarily in this order. While discussing the maximum possible product, it was pretty clear that 2 and 5 should be in the units' place, while 8 and 9 should be in the tens' place. That "the higher digits should be leading digits and the lower digits should be in the ones' place"

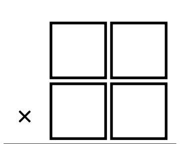

Figure 1

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is an observation that we elicit from such discussions. But which is larger: 95×82 or 92×85? How can we figure it out without actually calculating?

One player argued that 92×85 is larger since the gap $92 - 85 = 7$ is smaller (compared to 95×82 with the gap $95 - 82 = 13$). He argued that to maximize the product, the gap between the numbers should be minimized.

Is this true?

1. Can you check the following?
   a. $73 \times 52 =$ ________ vs $72 \times 53 =$ ________
   b. $61 \times 84 =$ ________ vs $64 \times 81 =$ ________
   c. $92 \times 41 =$ ________ vs $91 \times 42 =$ ________
   d. $85 \times 72 =$ ________ vs $82 \times 75 =$ ________
2. Does it generalize beyond 2-digits?
   a. $95 \times 3 =$ ________ vs $93 \times 5 =$ ________
   b. $84 \times 2 =$ ________ vs $82 \times 4 =$ ________
   c. $743 \times 12 =$ ________ vs $123 \times 74 =$ ________
   d. $854 \times 23 =$ ________ vs $234 \times 85 =$ ________
3. Does it generalize even when the largest digit is not a leading digit?
   a. $36 \times 4 =$ ________ vs $34 \times 6 =$ ________
   b. $59 \times 28 =$ ________ vs $58 \times 29 =$ ________
   c. $190 \times 46 =$ ________ vs $140 \times 96 =$ ________
4. $27 \times 35 =$ ________ with a gap of $35 - 27 =$ ________ vs $73 \times 52 =$ ________ with a gap of $73 - 52 = 21$? Why doesn't it work in this case?

The idea is borrowed from the known result that the area of a rectangle is maximized if it is a square. One can explore and see that in fact, as a rectangle gets closer and closer to a square, its area increases. Now a rectangle gets closer to a square if and only if the lengths of any pair of consecutive sides become more and more equal. Or in other words, if the gap between the lengths of two consecutive sides gets smaller and smaller.

But there is one more condition that must be fulfilled for this optimization to work. The perimeter of the rectangles must remain fixed as the lengths of the sides change.

How is that related to our problem?

Consider two rectangles:

	Rectangle A	Rectangle B
Dimension	95 cm × 82 cm	92 cm × 85 cm
Perimeter	2 (95 cm + 82 cm) = 2 × 177 cm	2 (92 cm + 85 cm) = 2 × 177 cm
Area	95 cm × 82 cm = 7790 cm^2	92 cm × 85 cm = 7820 cm^2

Since we had already fixed 2 and 5 as units, and 8 and 9 as tens, we get the same sum, i.e.,

$$\begin{aligned} 95 + 82 &= 90 + 5 + 80 + 2 \\ &= 90 + 2 + 80 + 5 \\ &= 92 + 85 \end{aligned}$$

using a combination of commutative and associative properties of addition.

This sum is nothing but half of the perimeters of the rectangles A and B. So, the condition of the fixed perimeter is met in this case. Now the area is nothing but the product of these numbers. Therefore, the area is maximized when the numbers are closer.

So, generally speaking, if the sum of the two numbers remains the same, then their product is maximized when their difference is the least.

Note: Is the sum remaining the same for the two pairs of numbers in Q4?

It is not easy to go beyond one topic. But this player observed that in a 2-digit × 2-digit multiplication game (in which the higher digits were placed in the tens place), the sum of the numbers remains the same. Therefore, he could connect it to the fixed perimeter condition. He also noted that the product is essentially the area of the rectangle. And thus used the known result for an area to maximize the product. A brilliant use of mensuration to solve a problem in arithmetic!!

Food for thought:

1. What would be the minimum product for 2, 5, 8, 9?
2. How would you strategize if you want to minimize the product for 2-digit × 2-digit?
3. How would you strategize to optimize the product for 2-digit × 3-digit?

The original idea of this family of games was to prevent students from cheating or copying from each other. However, it turned out to provide much more than that!

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