

Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

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The Point of Mathematics



5 Features

- » What is New in the New Class 1 and Class 2 Mathematics Textbooks?
- » Who Teaches Math Vocabulary?

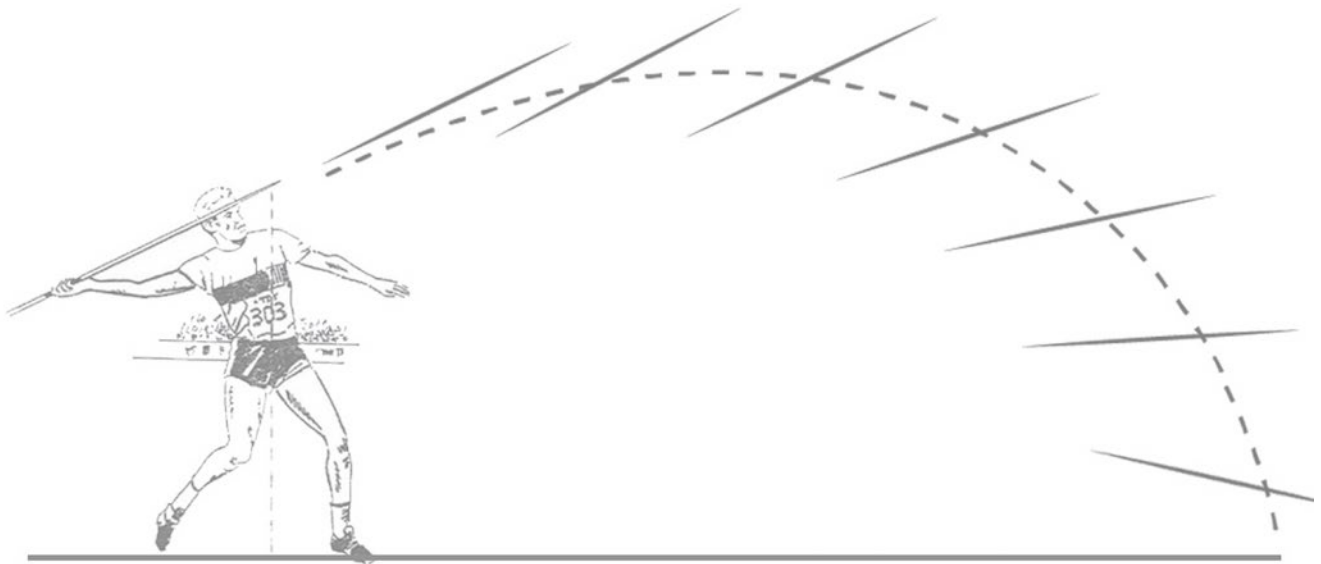
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PULLOUT
MASTERY OF
MULTIPLICATION



'But what is the point of this???' A question heard by many mathematics teachers as they struggle to convince their students about the 'usefulness' of the concepts and algorithms that they are learning. The ubiquity of mathematics in every activity of humans, makes this question both easy and difficult to answer. Easy because it is so obvious and difficult because it cannot be expressed in a nutshell. Take this image for example: We see shapes, angles, numbers, measurement, data, patterns, history and culture intertwined in this sport. Yet how conscious are we of mathematics when we strive to better ourselves?

From the Editor's Desk . . .

At Right Angles is part of the Azim Premji Foundation's endeavour to provide good-quality learning resources to practising teachers and teacher educators. Our aim is to promote dialogue among schoolteachers, practitioners and educationists on current discourses and perspectives on education, pedagogy, and on-the-ground experiences. The intent is to help build teacher capacity and facilitate more experiential and meaningful teaching-learning processes. In order to celebrate teaching with purpose and passion, At Right Angles showcases experiential and practical understanding grounded in the reality of India and the societies we live in.

As the November 2023 editorial described in some detail, the March 2024 issue of At Right Angles is significantly different from previous issues. So, it is only fitting at this point, to thank Shailesh Shirali for his active lead and guidance as Chief Editor for all our past issues. Our thanks also to the editorial committee members, K Subramaniam, Prithwijit De, Shashidhar Jagadeeshan, A Ramachandran and Jonaki Ghosh for their significant contributions into making the magazine what it is today. We will continue to look to them for their suggestions and input whenever possible.

Our primary audience is the primary and upper primary school teachers in the public education system. Hence the content of this and subsequent issues will be aimed at being of direct relevance to them and for Resource Persons on-the-ground, who work with teachers to help them teach better so that their students can learn better. With this in mind, we have taken some cues from the Mathematics section in the NCF 2023 also.

So, our **Features** section of this issue begins with Sandeep Diwakar describing *What's New in the New Mathematics Textbooks for Class 1 and Class 2*. Hriday Kant Dewan discusses how to address pedagogical, material and process challenges in the local context of teacher & school environment in his article on *Contextual Problems*. Rima Kaur's article *Who Teaches Math Vocabulary* probes the invisible separating lines between disciplines.

The **ClassRoom** section focuses on the concepts of multiplication and division. Arddhendu Dash shares his thoughts on the *Teaching Approach to the Division Algorithm* and Jitendra Verma describes and explains three ways to check the *divisibility of a number by 7*. Swati Sircar describes a game to *optimize the product* of two numbers and also explains the advantages of teaching *division by multi-digit divisors*. In all these articles, the process skills of visualisation, estimation, reasoning, and logic are emphasized.

The **Review** section continues where Classroom stops with a review of FLU- Flats, Longs and Units or the basic Base 10 number blocks.

In our new section **The Joy of Mathematics**, we celebrate the subject with two articles by Mohan R - *Javelin Throw* and *Guesstimation - the Fermi Estimation Problems*. How

does mathematics come into play in unseen ways? And in how many ways can a problem be solved - read the story about the hens and the rabbits to find out!

We close as usual with the **PullOut**- how do we check for *Mastery of Multiplication*? Padmapriya Shirali describes several ways- all of them providing drill and practice while developing higher-order thinking skills, none of them tedious or stressful.

As we settle into this new avatar, we hope to bring you more articles on Computational Thinking, some TearOuts with worksheets for your classes and of course, a complete Problem Corner. For now, enjoy the problems in the Fillers and don't hesitate to send in your solutions to AtRightAngles.editor@apu.edu.in

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At **Right Angles** is a publication of Azim Premji University. It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

Note: All views and opinions expressed in this issue are those of the authors and Azim Premji Foundation bears no responsibility for the same.

Contents

Features

Our premiere section presents a diverse range of articles centred around mathematics education. The range varies from discussions on various aspects of mathematics classroom practice to critical analysis of textbooks. Written by practising mathematics educators, the common thread is sharing critical perspectives.

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ClassRoom

Articles in this section are helpful in designing and implementing effective instruction. They draw directly on personal experiences of teaching with reflections on both successful plans as well as opportunities and ideas for improvement. They strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

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The Joy of Mathematics

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Review

Our Review section offers a diverse range of insights on educational resources in mathematics. We examine a variety of resources: textbooks, books on mathematics and mathematics education, teaching-learning materials, interesting websites, and educational games and software. We feature evaluations not only from experienced educators and subject experts but also welcome perspectives from practitioners and enthusiasts in the field. This inclusive approach ensures a rich variety of viewpoints, providing our readers with comprehensive and accessible reviews that can inform their choices in educational materials.

Math Space

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PullOut

The PullOut takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine.

Padmapriya Shirali

Mastery of Multiplication

What is New in the New Mathematics Textbooks?

SANDEEP DIWAKAR

The National Curriculum Framework 2020 recognized the importance of developing a strong foundation of learning during the early developmental phase (3 to 8 years). Keeping in mind the holistic development of children, the National Curriculum Framework – Foundational Stage, 2022 (NCF-FS, 2022) recommends curricular goals, competencies and learning outcomes associated with developmental domains such as physical, social, emotional, ethical, cognitive, language, aesthetic, cultural values and positive learning habits. Along with this, emphasis has been laid on the integration of all the domains while developing learning material including textbooks for mathematics.

“Textbooks help the teacher by providing an organized, sequential, consistent, and meaningful learning experience in order to achieve expected curricular goals, competencies, and learning outcomes. Textbooks also guide children and provide reliable reference points. Among the teaching materials used in the classroom, the textbook is one of the materials that plays an important role in planning classroom procedures, pedagogy and assessment.”

The National Curriculum Framework - Foundational Stage, 2022 has given the following suggestions regarding mathematics teaching methods:

1. Experience → Spoken Language → Pictures → Symbolic Language.
2. Linking mathematics learning to a child’s real life and prior knowledge.

3. Seeing mathematics as a means of problem solving
4. Engaging in mathematical communication using discussion and reasoning.
5. Developing a positive attitude towards learning mathematics.

In the light of the new policies, NCERT has developed new textbooks (Joyful Mathematics) for classes 1 and 2 in the year 2023. These books contain a number of activities that encourage organized work and learning within and outside the classroom with a focus on experiential learning for the all-round development of children. An attempt has also been made to integrate language and age-appropriate physical and mental development into the book through the context around the child as mathematics education cannot be separated from these. The book functions both as text-book and work-book and provides appropriate opportunities for children to learn, draw, colour and write through play.

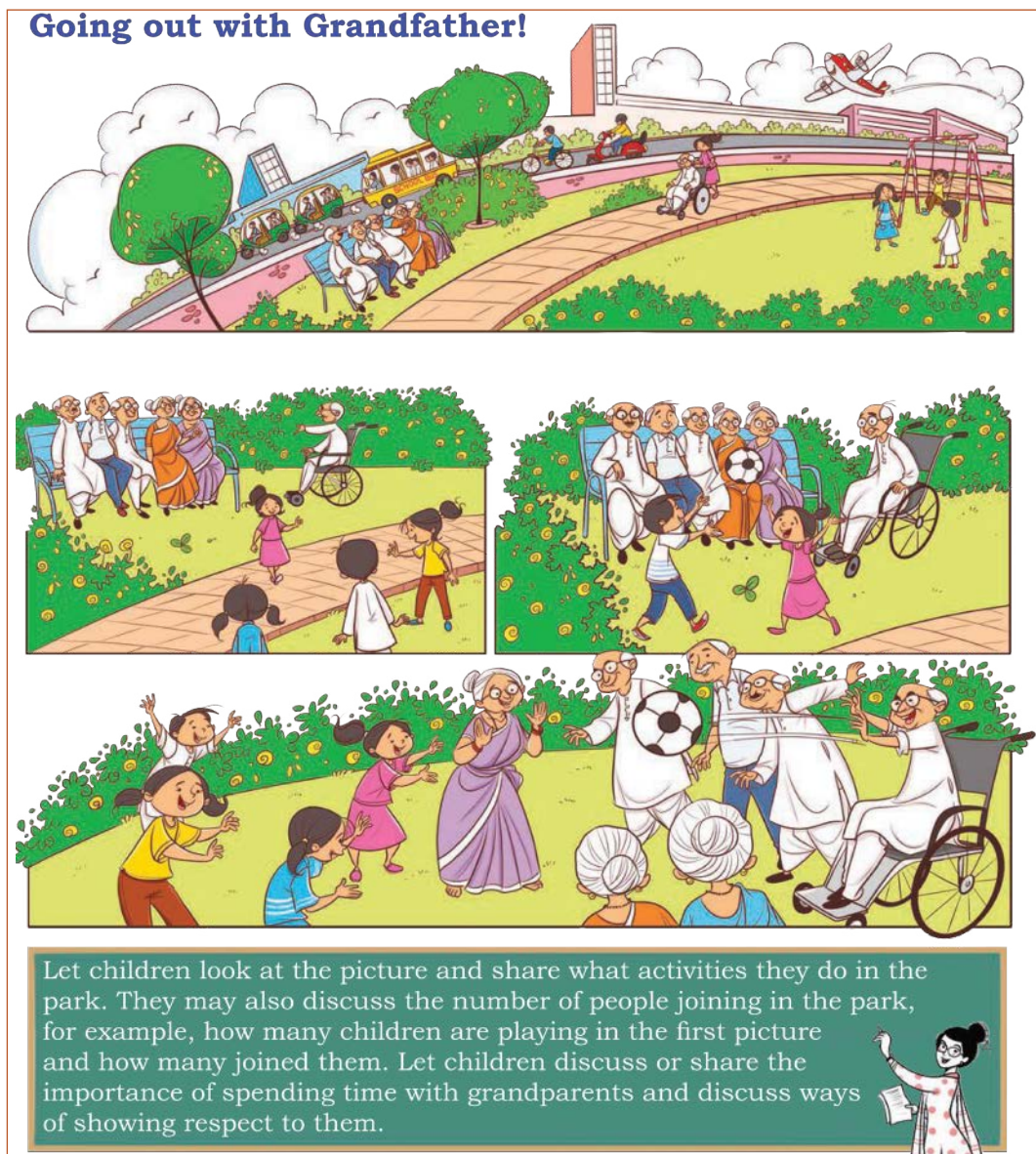


Figure 1. NCERT Textbook Class 1 Chapter 5 page 48

This beautiful sequence of pictures (Figure 1) subtly messages inclusion and relevance of the older generation and the importance of caring for them. At the same time, it provides counting practice for young ones.

Activity should be conducted in a manner so that all the children are engaged, irrespective of their differential abilities. For example, a *ghungroo* can be attached to the ball, and surface of the basket can be made different from the surface outside in order to get specific sound when the ball is in or out of the basket.



Figure 2. NCERT Textbook Class 1 Chapter 1 page 4. Suggestions for inclusion have been given in this and other chapters



Let us Talk

- A. Observe and discuss what is happening in the picture.
- B. What are the different things which you see in the given picture?
- C. How many houses are there?
- D. How many people are there in the picture?
- E. Guess how many lines are used to draw the leaves of a tree.
- F. Do you know the special name for this type of painting?
- G. Which part of India is famous for 'Warli' painting?

Figure 3. NCERT Textbook Class 1 Chapter 8 page 96.

Notice how art and culture are incorporated in this simple activity (Figure 3) which develops students' skills of observation, communication and reasoning and at the same time increases their general knowledge.



Project Work

- A. Collect pebbles, flowers, leaves, glasses, bowls, sticks, bangles, coins, caps, etc., and arrange them in a pattern. Create different patterns of jewellery, floral pot arrangements, art showpieces, etc.
- B. Observe and find the patterns in nature like leaves, butterfly, animal skins (cat, dog, zebra, tiger), curtains, *sarees*, *dupattas*, tiles, beehive, etc.
- C. Collect different objects seen around and make a collage.
- D. Create patterns using different actions like clapping, snapping your fingers, stamping your feet, etc.

The world around us is full of shapes. Encourage children to appreciate the rich heritage of India through exploring beautiful patterns in temples, mosque, church, gurudwara and monuments around them. Also ask them to share their observations of patterns in different art forms, movement patterns in dance forms, etc.




Figure 4. NCERT Textbook Class 1 Chapter 9 page 104. Children see math in the world around them while embracing diversity

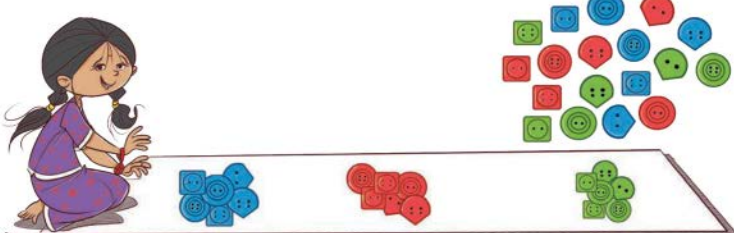
Many teachers had commented that the previous books were too wordy and that there were practical constraints in carrying out the given tasks in the classroom. Teachers have also been highlighting the lack of practice exercises in the old books. An attempt has been made to address these opinions. Most lessons begin with a poem, game, short story or activity related to the world around the child. For example, in the beginning of the Class 1 book, the cat hides here and there and sometimes it is seen above the window, sometimes under the bed, sometimes above the car, sometimes under the carpet. Similarly, to make sense of numerical concepts, the use of easily available materials such as pebbles, leaves, buttons, etc. have been suggested. Low-cost Teaching Learning Materials (TLMs) using locally available materials (cards, pieces of wood, fingers, buttons, ...) as well as the ginladi (number chain), dotted cards and number strips are also highlighted. There is a slant towards developing the skills of logical thinking, analysis and mathematical communication through activities, open-ended questions, explorations and discussion.



Figure 5. NCERT Textbook Class 2 Chapter 2 page 19. Building their understanding of shapes

 **Let us Do**


Suwali has placed all the buttons in three groups.



Why did Suwali make such groups?
Help Suwali to sort these buttons in other ways by drawing them.

Figure 6. NCERT Textbook Class 1 Chapter 1 page 9.

Such play-based activities (Figure 5 and Figure 6) allow students to develop a confident approach to mathematics.

 **Project Work**

Take ten cards 0 to 9.
Arrange the cards in such a manner that their sum must be 9.
There are many ways to do it. In how many ways can you do it?

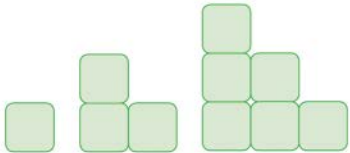
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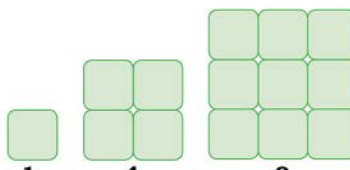
Figure 7. NCERT Textbook Class 1 Chapter 5 page 56.

The textbook has previously provided practice in writing the numerals and in the addition of single-digit numbers. In chapter 3, students have prepared their own number cards as a project, so materials are not a constraint for this project (Figure 7). Notice that it provides the opportunity to practise addition in a fun and open-ended manner. By comparing their answers, students get an opportunity to discuss and justify their strategies.

Exploring Patterns

Observe how the number of blocks are increasing in the following shapes and extend the pattern.

A.  _____

B.  _____

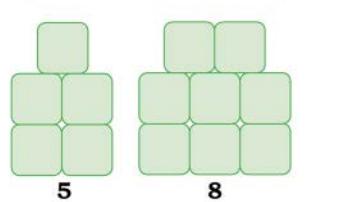
C.  _____

Figure 8. NCERT Textbook Class 2 Chapter 3 page 31.

Students with different learning styles will be able to absorb number patterns better by such visualisation (Figure 8) and it is hoped that teachers will also be able to provide hands-on learning using counters and building blocks inspired by such puzzles.

Play with Numbers

Who am I?

- A. I am the largest two-digit number.
- B. I am the largest two-digit number where no digit is repeated.
- C. I am the smallest two-digit number.
- D. I am the smallest two-digit number, my digits are repeated.
- E. I am the smallest two-digit number with 3 at the tens place.
- F. I am the largest two-digit number with 2 at the ones place.

Make your own such questions.

Figure 9. NCERT Textbook Class 2 Chapter 1 page 13.

Problem posing is an important indicator of conceptual understanding and the textbook showcases opportunities for this. See Figure 9 for an example.

The textbooks for both classes have puzzles at the end which give children the opportunity to apply their arguments and solve them. These riddles also hope that teachers can also make such puzzles and give children to solve. These textbooks prepared at the national level have tried to incorporate the diversity of the whole of India, but they also have their own limitations such as contexts which are familiar in one region may be alien to another. Therefore, instructions have also been given for teachers to include local games and toys and locally available material in classroom teaching. Naturally, it becomes necessary for the teachers to pre-plan the lessons thoroughly keeping in mind the teaching prompts given with them.

The previous set of textbooks was developed nearly sixteen years back with the same approach. However, it did not really translate these ideas into practical ideas that assisted in both pedagogy and assessment based on the concepts to be taught. While an approach similar to what is seen in the new textbooks was taken in the theme chapters, teachers did not really appreciate the point of these. Now that the goals of mathematics education have been distilled into the chapters of the textbooks for classes 1 and 2, it is hoped that teachers will be able to use this launching pad to enable *children to develop mathematical understanding and abilities to recognize the world through quantities, shapes, and measures* as described in the curricular goal in the NCF-FS.

Editor’s Note: All images from textbooks reproduced with permission from NCERT.

Reference:

1. National Council for Educational Research and Training (NCERT). (2023). *Joyful mathematics (Class 1 and Class 2)*. <https://ncert.nic.in/textbook.php?aejm1=11-13>



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Lights Off!

Let’s play a game! Picture a square-shaped house containing four square rooms, as illustrated in Figure 1.

Each room in the house is equipped with a light source. Each light can be controlled via a switchboard located outside the house. This house is designed so that if the light in room x is turned on or off, all rooms sharing a wall with room x will change their state: if they were originally on, they will turn off, and vice versa.

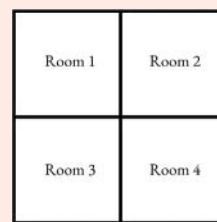


Figure 1

Upon your arrival at the house, assume that all the lights in all rooms are switched on. Then, you can proceed with the following steps to turn off all the lights.



Stop and Think! What problems can you pose based on this situation?

Look for our suggestions on Page 48

Filler contributed by Mohan. R

Who Teaches Math Vocabulary?

RIMA KAUR

Let us look at the following vignette.

Vignette 1

Christina has just begun class 1. During the term-end parent-teacher meeting, her class teacher (who is also her Math teacher) tells her parents – “*Christina is just not able to understand simple word problems, no matter how many times I solve them on the board. Please talk to the English teacher to sort out this issue, otherwise this will create many more problems in future.*” Slightly puzzled as Christina swiftly and correctly answers most addition and subtraction problems at home, be it $3+5$ or $7-3$, Christina’s mother closely examines her notebook and finds the following:

Jennifer skipped 8 times in the morning. She then skipped 3 times in the evening. How many times did she skip during the day?

Christina’s answer: 5

Lynpu goes to the river and catches 7 fish. His friend goes to the river and catches 3 fish. How many more fish does Lynpu catch?

Christina’s answer: 10

Prasida has 5 idlis on her plate. She eats 2 idlis. How many idlis does she have now in total?

Christina’s answer: 7

Keywords: Foundational Literacy and Numeracy, math vocabulary, Foundational Stage, early childhood education

Christina commits errors which we commonly associate with the initial stages of learning addition and subtraction. While errors are always useful for learning, they also present us with an opportunity to examine if there are any underlying pedagogical reasons which are leading to misconceptions in the minds of learners. Both from Christina's mother's experience and from the worksheet responses, it seems clear that Christina knows addition and subtraction – she simply does not know which to use when. The inability to solve word problems possibly stems from common practices related to the teaching of mathematics (Shirali, 2016) (10), which Christina's teacher may have followed too:

- Using confusing vocabulary while teaching a mathematical concept or encouraging rote-memorization of cue words e.g., 'more' and 'total' mean addition.
- Not addressing conflicts between everyday words and mathematical concepts. For example, here Christina possibly mistakes the word 'skip' to mean 'subtract'.
- Teaching addition and subtraction as mere procedures without developing important skills such as following instructions, visualization, and problem-solving.
- Seeing vocabulary development as a language problem and not as a mathematical concern.
- Designing poorly framed word problems, which are often not related to the children's context.
- Not 'thinking aloud' or demonstrating/verbalizing the thought process of solving mathematical problems so that children not only understand the steps better but also develop conceptual clarity.
- Not creating opportunities for children to share their thought processes.
- Underestimating the ability of children to formulate and solve their own mathematical problems.

The relationship between vocabulary and comprehension is intricate. Children who have a rich vocabulary are not only able to engage deeper with rich language contexts such as participating in conversations, playing, cracking jokes, and telling and listening to stories, but are also able to derive meaning from mathematical problems. As in the case of Christina, many learners face no real issue with computation but struggle to interpret and use language. Explicit instruction of vocabulary is shown to improve comprehension among learners in all subjects, including for children with disabilities (Powell & Driver, 2014) (9).

While teachers in the Foundational and even Preparatory Stage (from preschool to class 5) are recruited with the clear expectation that they will teach all subjects, there are some ground-level realities such as:

1. Teachers ill-prepared to teach all school subjects despite having the requisite pre-service qualifications.
2. Teachers teaching only one/two subjects to multiple classes i.e., different teachers teaching different subjects. This scenario is similar to Vignette 1 where the mathematics teacher shifts/transfers responsibility to the language teacher. This results in clear separation between subjects which is reflected in not only how time is organized (timetable) but also in how different subjects are taught – where no interconnections are built despite there being multiple, organic opportunities – especially in the Foundational and Preparatory Stage classrooms.
3. Teachers having varying levels of interest in different subjects. In other words, teachers taking more interest in teaching certain subjects and not others.
4. Fear of mathematics in teachers themselves, which transfers to the children.

What could Christina's teacher do instead?

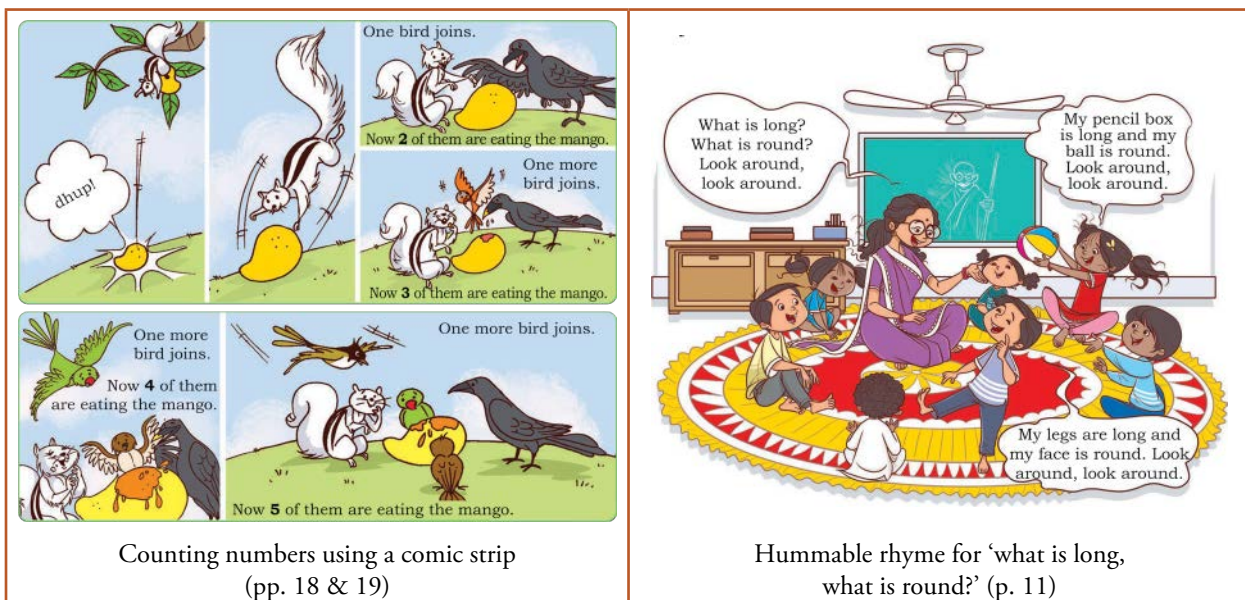
Christina's teacher will certainly benefit if she derives her pedagogy from the Curricular Goals and Competencies listed in the National Curriculum Framework for Foundational Stage (NCF-FS) 2022 (6). The Curricular Goal associated with Mathematics (CG-8, within the domain of cognitive development) expects learners to develop mathematical understanding and abilities to recognize the world through

quantities, shapes, and measures. Of the 13 Competencies corresponding to this Goal, the Competency associated with addition and subtraction (C-8.6) clearly states that children should be able to do so fluently and using flexible strategies. This Competency should be seen alongside another (C-8.13), which expects children to be able to formulate and solve simple mathematical problems. Moreover, there is a whole other Competency (C-8.12) on developing an adequate and appropriate vocabulary for comprehending and expressing concepts and procedures related to quantities, shapes, space, and measurements. However, CG-8 cannot be read in isolation. The previous Curricular Goal (CG-7) plays a critical role in helping children develop their understanding and abilities for mathematics, as it is about how children can observe the world around them and think logically to understand categories and their relationships.

The learning standards summarized above suggest a clear approach that the teacher needs to take while teaching Mathematics:

- Talking using everyday language to develop familiarity with quantities, shapes, and measures in the world.
- Identifying how real-life concrete experiences and exploration of the world can be integrated with the teaching of mathematical concepts to develop sound conceptual clarity.
- Integrating mathematics learning with other domains and not seeing it as separate concepts and skills to be developed in a separate time i.e., exploring opportunities for mathematics through stories, rhymes, games and sports, and art.
- Working closely with the language teacher (if different) to understand how all classrooms can support vocabulary development and comprehension in a comprehensive manner.

Given below are some samples from the new mathematics textbooks by NCERT for class 1 (8), where diverse contexts such as stories, rhymes, pictures, anecdotes, and comic strips are used to teach mathematics. Notice the use of language in the samples, and the richness of the contexts which are derived from children’s real-life experience. The textbooks, recently developed, are aligned with the Curricular Goals and Competencies defined in the NCF-FS 2022.



Strategies

Let us look at three specific strategies that are useful for strengthening mathematics related vocabulary in the Foundational Stage. These strategies are also reflected in the samples from the textbook shown above. As discussed earlier, these can be used by all teachers and in different periods/blocks in the timetable. Please note that these are in addition to the mathematical games of which you can find many examples.

Reading/ talking about pictures

Discussing pictures can enhance mathematical thinking by encouraging students to visually analyze and interpret the depicted information, on which further problems can be built based on the creativity of the teacher. For example, the picture below from *'Market Mayhem'* by Soumya Menon (4) provides a rich context for many mathematical concepts already discussed above. Can you think of how you would use this picture in your classroom?



Children's literature

When selecting children's literature for teaching mathematics, prioritize stories/anecdotes that naturally incorporate mathematical concepts. Look for opportunities to integrate words from children's daily lives, making mathematics more relatable and enhancing comprehension through familiar contexts. For example, the Nepali story embedded in a learning experience *'Kun Lamo, Kun Choto?' (Which is Long, Which is Short?)* in the Draft Sikkim Preschool Curriculum (SCERT, 2024) (11) is about a lone stick in the woods that hops around and meets more sticks of varying lengths, which become its arms and legs. The teacher tells the story and simultaneously builds a stick figure using real sticks collected by children from the outdoors. Follow-up activities involve children making their own stick figures and comparing the lengths of the sticks they have used for the body, arms, and legs. The teacher uses words in Nepali like *'lamo'* and *'choto'* but also introduces English words alongside.

Here is another example of a story called *'Gulli's Box of Things'* by Anupama Ajinkya Apte (1), where words such as 'small', 'big', 'wide', 'top' are used while describing the assortment of objects collected by Gulli in his little box.

“What is wrong, Mangal Chacha?” asked Gulli one Sunday.

“I have to pour oil from this pack, but the bottle's neck is so small. I am sure I will make a complete mess in my kitchen,” said Mangal Chacha.

“Ah, don't worry, let me find you something really fast,” said Gulli.

Clink, clonk, dadum-dum, let's see what Gulli has found this time!

A funnel with a big, wide mouth! Pour anything from the top, and see how it comes down so neatly, drop by drop!

This third sample story ‘*The Very Shocking Report Card*’ by Jane De Suza (5) is about a boy, Pattu, who dreads receiving his report card because he knows he will not score too well. The story integrates mathematical and socio-emotional and ethical development. It also presents the opportunity for making simple calculations such as addition and subtraction.

So this time, Papa went with Pattu to get his report card from the teacher himself. Papa looked at Pattu’s report card. His smile went from a slice of papaya to the thin wedge of a lime.

3 out of 10 for Reading. Papa’s eyebrows dashed together like colliding bulls.

4 out of 10 for Recitation. He shook his head from side to side, like a tree in a storm.

2 out of 10 for Spelling. He let out a long whoosh, which sounded like a train engine.

On their way home, Pattu encounters different people in his neighbourhood and through their conversations, Papa realizes that Pattu is a thoughtful child who looks out for everyone. Once home, Pattu is scared and embarrassed to share his marks with his mother. But Papa shares a different report card...

“9 out of 10 for sharing,” said Papa.

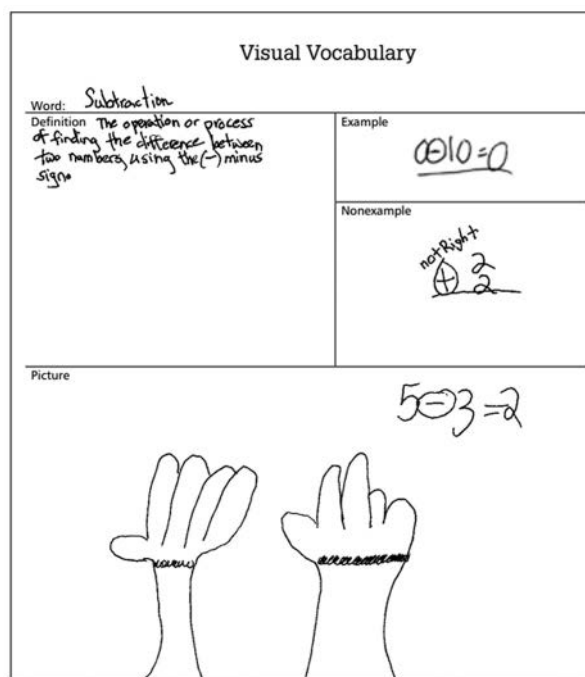
“10 out of 10 for kindness.

11 out of 10 for respect.”

Graphic organizers

Graphic organizers are visual depictions of content in the form of diagrams, tables, maps, etc. They are a novel and appealing tool for children to develop and showcase deep conceptual understanding by using context-based vocabulary (Bay-Williams & Livers, 2009) (2). Graphic depictions, particularly by young children in the Foundational Stage, make vocabulary visible and contribute to comprehension in the same way as reading pictures. In a study conducted in 1997, learners who received vocabulary instruction through graphic organizers significantly outperformed learners who received vocabulary instruction through definitions (Monre & Pendergrass, 1997, as cited in Powell & Driver, 2014) (9).

In the given graphic organizer (Bruun et al., 2015) (3), the child works on the concept of subtraction by defining it, formulating their own mathematical problems with an example and non-example, and designing a picture showcasing subtraction. The teacher can use this organizer to explore the child’s reasoning (*Why does the child give ‘ $0 - 10 = 0$ ’ as an example? Did the child intend to write ‘ $10 - 10 = 0$ ’, or ‘ $10 - 0 = 0$ ’, or is there any other reason?*); identify misconceptions (*In the picture, the example ‘ $5 - 3 = 2$ ’ is correct, but why does the picture show two folded fingers and three open fingers? What is the child’s process of arriving at the answer?*); and obtain insights on the teaching plan (*Should definitions be taught and expected to be reproduced? Can children arrive at their own definitions?*).



Summary

In the Foundational Stage, basic pre-numeracy concepts related to counting, identifying numbers, and comparing quantities enable children to develop successful computational abilities such as adding and subtracting, developing a sense of basic shapes and measurement, and early mathematical thinking. Misconceptions arising from the use of language can impede mathematical comprehension. Explicit vocabulary instruction using content related to children's real-life experiences and diverse strategies that provide multiple exposures (the key word being 'multiple', and not 'repeated!') need to be woven into the daily schedule and not exclusively left to a single teacher or period/class. Deriving an approach in alignment with the Curricular Goals and Competencies can help learners progress towards the key curricular aims of mathematics education given in the NCF for School Education (NCF-SE, 2023) (6) i.e., basic numeracy; mathematical thinking; problem solving; mathematical intuition; and joy, curiosity, and wonder!

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Contextual Problems in Math: The NCF-SE Approach

HRIDAY KANT DEWAN

Policy and action conversations and programs have been attempting to address the lack of learning and capability in mathematics. As a parallel process to this, what mathematics to teach and how it should be taught has been a matter of concern, ever since this started being emphasized in curricular frameworks. What is to be understood as important mathematics is evolving - particularly in the context of the teaching of mathematics. Given that easy-to-use calculators are accessible now, does foundational mathematics education need to focus exclusively on rote memorization of algorithms or calculation tricks?

According to the NCF-SE 2023, *The ability to formulate problems, develop many alternative solutions, evaluate different solutions to choose the most optimal solution, and implement the solution is again indispensable in achieving all five Aims. Problems that require quantitative models require the mastery of various mathematical procedures, starting from simple arithmetic skills of addition and subtraction to more complex solving of algebraic equations. The use of computational models for solving problems would require computational skills. Skills for logical reasoning include constructing and evaluating arguments, both formally and informally.*

It is with this perspective and much more that we look at the emphasis on the problems chosen for the student to develop these capacities. Contextually relevant problems are not simply word problems which are presented with attractive figures. They


Keywords: Word problems, context, sense-making

should, of course, be derived from children's life experiences. At the same time, the child should be able to see the need to solve the problem and understand how these solutions can impact their daily routines and their decision-making.


The usual mathematics class does not give the student such opportunities and in almost all schools, there is no time for children and teachers to savour mathematics and problem solving. Neither is time allotted for relaxed conversations

about the concepts that are to be transacted nor are these concepts linked to the context of the children. For example, a question from the Math Magic series (Figure 1), (designed to make children work with mathematics from within their context, but often transacted mechanically), focuses on a currency that is no longer in use in India, making it difficult for children to relate to it unless it is part of a discussion on larger issues about conversations with grandparents and rising costs of living.

Rupees and Paise

How many  will make one rupee?

Is 50 paise half of one rupee?

How many  will make one rupee?

25 paise is _____ part of one rupee

20 paise is _____ part of one rupee

How many 10 paise will make one rupee?

So 10 paise is _____ part of one rupee.

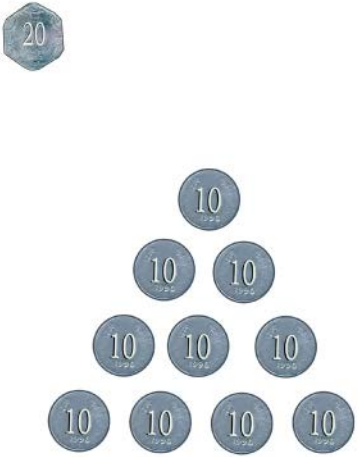


Figure 1. NCERT Textbook Class 5 Chapter 4 page 65.

Secondly, no opportunity is given to students to unravel the nature of the problem and extract from it the information that is relevant to the problem. Only then can they comprehend the intention of the question and how to go from the information available to the one that is sought. The effort is towards building the ability to follow algorithms stepwise and remember the steps like a mechanical drill. The message is 'do not deviate from the steps, follow them precisely'. The advice is to not use your conceptual understanding and follow a different method of solution, as you may make a mistake. The classroom processes, and the so-called TLM (Teaching Learning Materials),

are very often geared towards this. In a sense, they are expected to present to the learners the 'concrete' steps of the algorithm and help them remember them. They neither try to explain the process and the steps nor allow children to use alternative strategies at all. The problems are also designed so as to emphasise the practice of the algorithms mechanically. For example, this page (Figure 2) in the Class 5 NCERT Textbook (even though it is only focused on the breaking up of the algorithm of multiplication and its piecemeal practice) can have plenty of scope for the teacher to discuss the 'why' of the algorithm, but how often is this done?

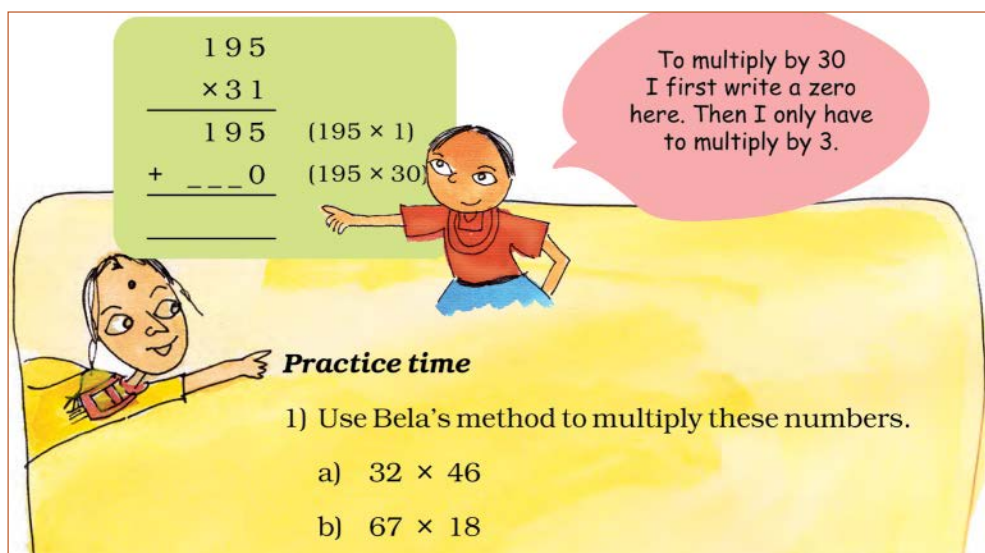


Figure 2. NCERT Textbook Class 5 Chapter 13 page 171.

Amit Kulshrestha is a mathematics teacher and researcher, both as his profession and for pleasure. Given below is a summary of his article in Pathashala Bithar Bahar [1] and a subsequent webinar [2] on the same.

Very often in a hurry to give students ways to solve word problems, teachers suggest that they look for keywords to help decide what operations to use. He argues that the whole focus in the classroom is actually on giving children practice in extracting the correct numbers, applying the correct algorithms and memorising the known standard rules for solving questions. The classroom is focused on giving the explanation and the steps to follow to reach the answer. There is no space for allowing children to think and develop their own strategies and approaches to problems given. However, the national curricular documents including the current NCF forcefully argue that children must not just be *allowed* to but **encouraged** to develop and find their own strategies to solve problems. They must learn to appreciate that there can be many strategies to solve them and sometimes, many answers too. The NCFSE (2023) also suggests that the classrooms encourage multiple methods so that students develop their own strategies to evolve. The NCERT textbooks for mathematics as well as the NCFSE emphasise the need for children to be able to create problems.

Here are some examples which illustrate that questions are often worded in a manner and comprise situations that are not natural for the context of the children.

1. 114 birds were sitting in a tree. 21 more birds flew up to the tree. How many birds were there altogether in the tree?

It is unlikely for anyone to be able to count birds as they fly up to the tree. So, the context for the question seems more for form, than for any real pre-experience that the children can relate to.

2. Jane has 63 m of ribbon. If she cuts 56 m 21 cm ribbon from it, what length of ribbon will be left?

Even if we change the name (as is often unfortunately the solution to bringing in local context,) the numbers in the question make no sense. A ribbon of 63 meters is not generally heard of and then cutting a length of 56 m and 21 cm also makes it unnatural.

3. In a grocery shop, there was 2510 kg 350 g of wheat in the morning. During the day, 890 kg 600 g of wheat was sold out. How much wheat was left in the shop in the evening?

This is a surprising amount of wheat in stock and an equally surprising amount sold in the

day! Normal shops would usually keep stock in number of bags, rather than in grams.

4. Vishal wants to make a book tower of height 48 cm. If the thickness of each book is 12 mm, how many books will he need to make the desired height?
5. A box of frozen vegetables weighing 144 kg 780 g was delivered at a grocery shop. If there were 15 bags of equal weight inside the box, what was the weight of each bag?

Both these problems are supposed to be contextual questions. But it is clear that the objective is for students to practice dividing one number by another. Surely a thoughtful student would wonder who would want to make a tower of a specific height? Even if that is required, why would someone have so many books all of the same height and need to find the number of books in the stack? Why would such an accurate weighing of the box of vegetables be needed?

6. The length, breadth and height of a room are 24, 18, and 12 feet respectively. What is the longest tape that can be used to measure these?

The point about this is that the common experience of the child when they see a measuring tape being used is that a very long tape may be used to measure small distances. We need a long tape if we do not want to lift the tape and continue the measurement from the end point of the previous placement of the tape. Otherwise, we can use any tape and measure the length. So, we could use any tape above 24 feet in length, including the one that is 24 feet long. The intention of those who designed the question was to test if the child could find the Highest Common Factor of 24, 18 and 12 and use that as the longest tape so that there is no part measurement needed. The wording of the question does not match its objective. The language of the question may sometimes be ambiguous, but the answer expected does not allow for multiple options.

Clearly, the idea here is to get children to do some operations and that too with some specific kind of numbers and the context is just a false pretence

that does not add any value to their deriving an understanding of the concept or the notion of numbers. It does not give them procedural clarity either, as they cannot make sense of the number they get as an answer and have no way of knowing if it is even approximately correct.

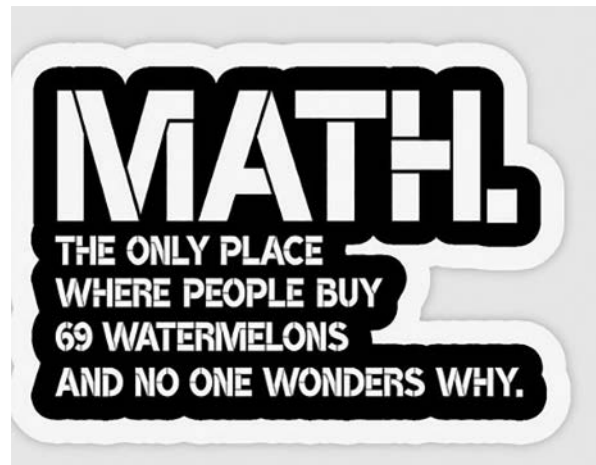


Figure 3

While all these may be called exercises for understanding the language of the question, they may not help in the aspects of relating mathematics to life and seeing connections. In fact, these problems are often even more contrived with complicated numbers and tedious use of algorithms. Very often it is not logically obvious how the solution has to be reached and it becomes necessary for the teacher to outline the solution steps. This is then memorized by the student. But they are not clear why the approach needs to follow the specific method suggested and why it leads to the answer.

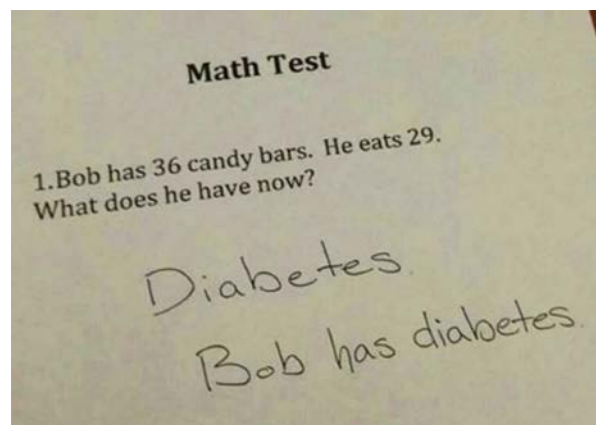


Figure 4

Here are some suggested questions which may relate better to the child's context.

1. Suresh uses a bucket and mug for bathing every morning. He saw that he got 12 full mugs of water from the filled bucket. However, one day, he found that he got only 9 full mugs of water. He realised that the mug was new. What would be the difference in the new mug with respect to the old?

This can relate to filling water in different containers and may lead to interesting conversations bringing in other experiences and also linking capacity to ratios and fractions.

2. Using square tiles how many tiles of different sizes can you use to fill a square floor of area 144 sq.m?
3. Could any rectangular tiles also tessellate the same floor area of 144 sq.m? What sizes of rectangles and how many are needed in each case?
4. Given the area of a floor is a^2 sq.m, where 'a' is a whole number, what are the possible sizes of square tiles that can fill the surface?

Note that the questions are getting more complex (and more interesting) at every stage. If we attempt these problems ourselves, we can see that it leads to counting the combinations and that in itself has opportunities to explore. Or one could go beyond that and try to find a way to solve it in a manner that allows for a general formulation for the possible combinations of numbers or the sizes of the square tiles.

Questions which enable students to use mathematical objects and concepts with understanding and develop curiosity and a sense of adventure are the need of the hour. A good guiding question while such questions are being developed is: What is the objective of giving the exercises and how do we view the nature of mathematics and

the learning of the same? What does the learner have to do to solve the question? The objectives for a task arise from our understanding of the foundations of mathematics and how we visualise the path of learning. And that has to show up in what the learner has to do in the task that you have given the learner.

The kind of problems around which contextual conversations can be facilitated are at the moment, rare if not entirely absent. Admittedly, these are not easy to formulate, also there is not enough time that can be allocated for discussing these. The concern about assessment and hence the entire process of mathematics teaching and learning being focused on memorisation, algorithm focus and lack of understanding and any real capability to explore and feel comfortable with mathematics persists. NCFSE 2023 points out that “*Most of the assessment techniques and questions focus on facts, procedures, and memorisation of formulas.*” However, assessment should focus on understanding, reasoning, and when and how a mathematical technique is to be used in different contexts. This is not a new thought and has been expressed since the beginning of the century and in some spaces of India even earlier, but this has been extremely difficult to find a way to start to put this in place. If this is done, then the aims of mathematics education as defined in NCF-SE will have a far greater chance of being met.

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Can Javelin Throws be Measured with a Broken Tape?

On the day of the school sports meet, the physical education teacher notices that the broken measuring tape hasn't been replaced. The javelin throw event is about to begin in 30 minutes, and there's no time to get a new tape. They need to measure the distance from the centre of the sector subtending the throwing arc to where the javelin lands (Figure 1). The issue is that the damaged tape can only measure up to 12 metres, while javelin throws typically cover distances of 10 to 60 metres.

She changes the field layout by drawing circular arcs around point O using ropes, with a 10-metre difference in radii (Figure 2). This way, she thinks that she only has to measure from where the javelin lands to the nearest arc.

The event takes place with this adjusted layout. However, by the end of the day, the runner-up questions the fairness of the teacher's method. He wonders how the point on the nearest arc is selected and questions the validity of the approach.

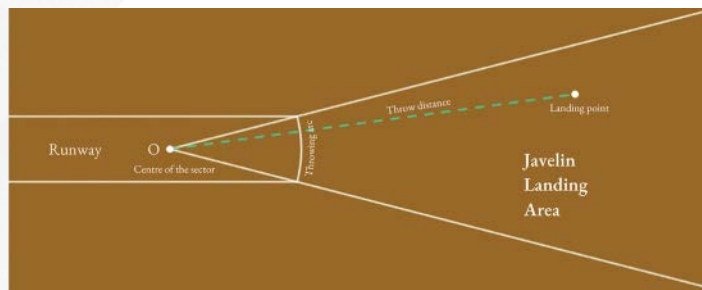


Figure 1. Layout for Javelin throw event

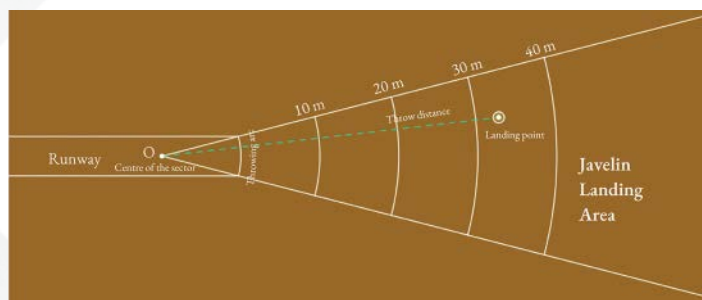


Figure 2. Modified layout

Question.

If the method used by the physical education teacher is mathematically sound, can you, on her behalf, describe the method of choosing the point on the arc and justify that the method is fair and sound?

For a solution turn to page number 59

Optimizing A Product

A Brilliant Use of Some Lateral Thinking!

SWATI SIRCAR

The game “Random Digits” demands Higher Order Thinking Skills (HOTS). The game starts with a common board for all players. The board is essentially for addition, subtraction or multiplication with two multi-digit whole numbers. It specifies the operations and the size, i.e., the number of digits of each whole number used in the operation. However, the exact digits are left blank. As the facilitator names each digit, players have to immediately place them on the board. Once placed, the position of a digit cannot be changed. Zero cannot be a leading digit, i.e., it cannot be placed in the leftmost box of any number. If a player is forced to do that, then s/he is disqualified. If the players choose to maximize the sum, difference or product, then the winner is the one with the maximum result (sum, difference or product). However, the players can choose to aim for the minimum as well. In both cases, players have to think where to place each digit in order to optimize his/her result. It is always a good idea to discuss the optimum result after each game. More details on the game can be found at <https://shorturl.at/hkxV3>

We were playing this game for a 2-digit \times 2-digit multiplication and wanted to maximize the product. The digits given were 2, 5, 8 and 9, not necessarily in this order. While discussing the maximum possible product, it

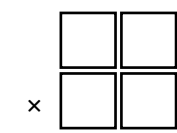


Figure 1

was pretty clear that 2 and 5 should be in the units' place, while 8 and 9 should be in the tens' place. That “the higher digits should be leading digits and the lower digits should be in the ones' place”

Keywords: Reasoning, making connections, strategizing, math games

is an observation that we elicit from such discussions. But which is larger: 95×82 or 92×85 ? How can we figure it out without actually calculating?

One player argued that 92×85 is larger since the gap $92 - 85 = 7$ is smaller (compared to 95×82 with the gap $95 - 82 = 13$). He argued that to maximize the product, the gap between the numbers should be minimized.

Is this true?

1. Can you check the following?
 - a. $73 \times 52 = \underline{\hspace{2cm}}$ vs $72 \times 53 = \underline{\hspace{2cm}}$
 - b. $61 \times 84 = \underline{\hspace{2cm}}$ vs $64 \times 81 = \underline{\hspace{2cm}}$
 - c. $92 \times 41 = \underline{\hspace{2cm}}$ vs $91 \times 42 = \underline{\hspace{2cm}}$
 - d. $85 \times 72 = \underline{\hspace{2cm}}$ vs $82 \times 75 = \underline{\hspace{2cm}}$
2. Does it generalize beyond 2-digits?
 - a. $95 \times 3 = \underline{\hspace{2cm}}$ vs $93 \times 5 = \underline{\hspace{2cm}}$
 - b. $84 \times 2 = \underline{\hspace{2cm}}$ vs $82 \times 4 = \underline{\hspace{2cm}}$
 - c. $743 \times 12 = \underline{\hspace{2cm}}$ vs $123 \times 74 = \underline{\hspace{2cm}}$
 - d. $854 \times 23 = \underline{\hspace{2cm}}$ vs $234 \times 85 = \underline{\hspace{2cm}}$
3. Does it generalize even when the largest digit is not a leading digit?
 - a. $36 \times 4 = \underline{\hspace{2cm}}$ vs $34 \times 6 = \underline{\hspace{2cm}}$
 - b. $59 \times 28 = \underline{\hspace{2cm}}$ vs $58 \times 29 = \underline{\hspace{2cm}}$
 - c. $190 \times 46 = \underline{\hspace{2cm}}$ vs $140 \times 96 = \underline{\hspace{2cm}}$
4. $27 \times 35 = \underline{\hspace{2cm}}$ with a gap of $35 - 27 = \underline{\hspace{2cm}}$ vs $73 \times 52 = \underline{\hspace{2cm}}$ with a gap of $73 - 52 = 21$? Why doesn't it work in this case?

The idea is borrowed from the known result that the area of a rectangle is maximized if it is a square. One can explore and see that in fact, as a rectangle gets closer and closer to a square, its area increases. Now a rectangle gets closer to a square if and only if the lengths of any pair of consecutive sides become more and more equal. Or in other words, if the gap between the lengths of two consecutive sides gets smaller and smaller.

But there is one more condition that must be fulfilled for this optimization to work. The perimeter of the rectangles must remain fixed as the lengths of the sides change.

How is that related to our problem?

Consider two rectangles:

	Rectangle A	Rectangle B
Dimension	95 cm \times 82 cm	92 cm \times 85 cm
Perimeter	2 (95 cm + 82 cm) = 2 \times 177 cm	2 (92 cm + 85 cm) = 2 \times 177 cm
Area	95 cm \times 82 cm = 7790 cm ²	92 cm \times 85 cm = 7820 cm ²

Since we had already fixed 2 and 5 as units, and 8 and 9 as tens, we get the same sum, i.e.,

$$\begin{aligned}95 + 82 &= 90 + 5 + 80 + 2 \\ &= 90 + 2 + 80 + 5 \\ &= 92 + 85\end{aligned}$$

using a combination of commutative and associative properties of addition.

This sum is nothing but half of the perimeters of the rectangles A and B. So, the condition of the fixed perimeter is met in this case. Now the area is nothing but the product of these numbers. Therefore, the area is maximized when the numbers are closer.

So, generally speaking, if the sum of the two numbers remains the same, then their product is maximized when their difference is the least.

Note: Is the sum remaining the same for the two pairs of numbers in Q4?

It is not easy to go beyond one topic. But this player observed that in a 2-digit \times 2-digit multiplication game (in which the higher digits were placed in the tens place), the sum of the numbers remains the same. Therefore, he could connect it to the fixed perimeter condition. He also noted that the product is essentially the area of the rectangle. And thus used the known result for an area to maximize the product. A brilliant use of mensuration to solve a problem in arithmetic!!

Food for thought:

1. What would be the minimum product for 2, 5, 8, 9?
2. How would you strategize if you want to minimize the product for 2-digit \times 2-digit?
3. How would you strategize to optimize the product for 2-digit \times 3-digit?

The original idea of this family of games was to prevent students from cheating or copying from each other. However, it turned out to provide much more than that!



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Thoughts on the Teaching Approach to the Division Algorithm

ARDDHENDU
SHEKHAR DASH

The National Policy on Education (NEP - 2020) highlights a severe learning crisis in basic mathematical skills, as evidenced by various governmental and non-governmental surveys. Why is this severe? When students fall behind on basic mathematics skills, they tend to maintain flat learning curves for years, unable to catch up forever. For many students, this has become a major reason for not attending school, or for dropping out altogether.

Division is an important topic in mathematics, but unfortunately, the majority of primary school children struggle with it, often making mistakes. This article primarily focuses on the types of mistakes made by students while performing the division algorithm, possible reasons for these mistakes and the suggested pedagogy to address them. It also highlights the importance of estimating the quotient, verifying the result and understanding the concept of division in different contexts in solving word problems.

Let's start with a question: if a child performs the division algorithm, can she check if the quotient is correct? The answer is yes, it is possible through estimation and verification of the quotient. However, in most of our classrooms, the teaching does not focus much on estimation and verification.

Estimation in mathematics is a process of rough calculation of an approximate answer in order to check for accuracy. This requires a high level of thinking skills. Estimating the quotient

Keywords: Procedural understanding, conceptual understanding, division algorithm, TLM.

is an important part of teaching division, where students can use this skill to get the approximate answer to the division problem and can also check the correctness of any answer. One way of estimating the quotient is to round the dividend and the divisor. Let's understand the estimation of quotient in the division problem $242 \div 22$. Rounding the dividend 242 and the divisor 22 to the nearest tens, we get 240 and 20. So, the mental calculation would be $240 \div 20$ which is $24 \div 2 = 12$. So, the estimated quotient of dividing 242 by 22 would be 12.¹ To do this, the student must be adept in rounding, in dividing by powers of ten and in the tables.

Another important aspect of division is the **verification of the result**. While teaching division, we must ask students to find the relationship between Dividend, Divisor, Quotient and Remainder. This relationship can be found by observing the relationship in different division problems and based on the pattern, students can find the relationship as "Dividend = Divisor \times Quotient + Remainder". And they will be encouraged to verify the quotient using this relationship. Suppose that, while dividing 517 by 5, a student gets the quotient 13 and the remainder 2. The student can verify the quotient like this:-

$$\begin{aligned}\text{Divisor} \times \text{Quotient} + \text{Remainder} &= 5 \times 13 + 2 \\ &= 65 + 2 = 67\end{aligned}$$

Which is not equal to the quotient 517. So, the student is alerted that the quotient is not correct.

Importance of understanding the context of division: For students, the challenge in solving word problems is understanding which number operation is to be used. Most of the students try to take hints from cue words in the question and use the operation related to that word. But it is not always necessary that the word indicates the operation and also to point out which numbers are involved. Let's understand from the following two examples.

Example 1: If 40 cakes are kept equally in 4 bags, then how many cakes will there be in each bag?

Example 2: Rajesh bakes 40 cakes and stores them in boxes of 10. How many boxes does he need?

In Example 1, the action verb "equally" indicates that division is needed to solve the problem, while in Example 2, the student needs to understand the question to solve the problem as there is no such cue word given in the question. So, it is required to understand the different contexts of the concept of division and as teachers, we need to discuss some of these contexts while teaching division.

Two contexts used to teach division in primary classes relate to "equal sharing" and "equal grouping".

Equal Sharing: In this context, we need to find out how much each portion contains when a given quantity is shared out into a number of equal portions. For example, if there are 6 mangoes in a basket and these are distributed among 3 students. How many mangoes will each student get? The simplest way to build an understanding of this context is to distribute one mango at a time to each student until all the mangoes have been shared equally.

Equal Grouping: This is the context in which we need to find the number of portions of a given size which can be obtained from a given quantity. For example, if there are 6 mangoes in a basket and we are making packs of 2 mangoes, how many packs will we make? This question is all about finding groups of 2 mangoes from 6 mangoes. This can be done through repeated subtraction.

¹ Note: The example used here is $242 \div 22$. When 242 is rounded to 240 and 22 to 20, the answer is quite close to the actual answer. But this may not also be the case. For example, when dividing 242 by 16, 16 also rounds to 20, giving an approximate answer of 12, but $242 \div 16$ is approximately 15. What is significant is that an approximate range in which the answer falls may be obtained. Please see the article on Multi-Digit Divisors in which estimation is discussed in greater detail.

Mistakes in carrying out the division algorithm: I have noticed that students make mistakes while using the division algorithm due to a lack of understanding of the concepts of division, subtraction, multiplication and place value. As mentioned earlier, one can verify that the answer of the division of $416 \div 4$ cannot be 14 as sometimes obtained due to leaving out the 'zero' in the quotient. This is checked by the fact that $14 \times 4 = 56$ which is not equal to the dividend i.e., 416 (Refer Figure 2A). Or the student can estimate that 14 could not be the quotient as when we divide $400 \div 4$ the quotient is 100, so the quotient in the question should be more than 100.

I had assigned a few questions on the division of whole numbers in class 4. I analyzed the students' responses on two points - what children know and what they need to understand. Let us go through a few sample answers.

In the division question $18 \div 7$, the first answer (Figure 1) indicates that the student knows the process of division but does not understand it completely. Here, the student does not know when all the places have been divided and whether to perform one more step or not. Also, the result in the quotient reflects that the student does not know the property of division – that for whole numbers, the quotient will be less than the dividend. Estimation can also be used to verify the answer.

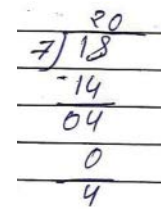


Figure 1

In the question $416 \div 4$, I present two responses. The first answer (Figure 2 A) indicates that the student has not divided one place at a time. When 4 hundred is divided by 4, the quotient has 1 in the hundreds place but then the tens and units place were combined to form 16 units, which was then divided by 4, to give 4 as the units digit in the quotient. Here the child could not write the quotient based on the place value system. Instead of 104, the child considered it to be 14. (In the case of the second response (Figure 2 B), we see that the child made a mistake in the multiplication of 4 by 0. This is an error commonly made by students and usually because, when teaching the tables, we start with multiplication by 1 instead of by 0. Later, this student ignored the 6 in the unit place of the dividend and assumed that the division was complete. In both these cases, estimation of the quotient would have helped the student.

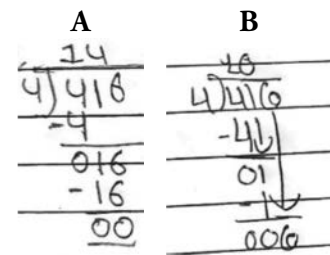


Figure 2

In the question $835 \div 8$, we can find that in the first answer (Figure 3 A), the child made a mistake in not accounting for the fact that after 8 it would only be 3 tens that 8 has to divide which should have given 0 in the tens place of the quotient. Instead, the student has divided 35 units by 8. Whereas in the case of the second answer, (Figure 3 B), the child could write the quotient when dividing 35 units by 8.

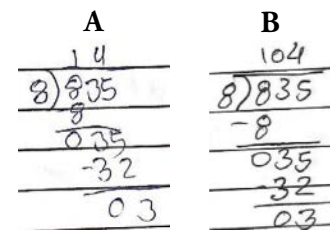


Figure 3

The difference between the two responses shown in Figure 4, makes a strong case for verification of the quotient. Figure 4B shows the correct process and quotient and the child has followed step by step division of each place. The student whose response is shown in Figure 4A may be encouraged to do this:-

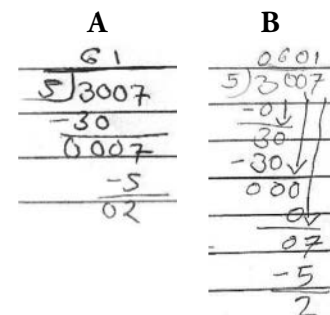


Figure 4

$$\begin{aligned} \text{Divisor} \times \text{Quotient} + \text{Remainder} &= 5 \times 61 + 2 \\ &= 305 + 2 = 307 \end{aligned}$$

Which is not equal to the quotient 3007. So, the student is alerted that the quotient is not correct.

In the division of $6359 \div 4$, the child made a mistake (Figure 5) perhaps because of not writing the digits in proper places i.e. units under units, tens under tens etc. Because of the displaced placement, the students may have missed the digit 5 of the tens place. These types of mistakes are generally seen in the case of the division of numbers having four or more digits. This suggests division involving larger numbers should be practised with adequate emphasis on estimating the anticipated answer before starting the formal algorithmic process.

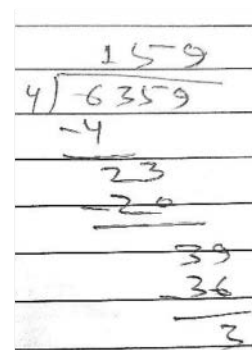


Figure 5

From the above answers, we can conclude that basic mistakes made by children are –

- Understanding of place value,
- Multiplication by zero. Some students assume that $4 \times 0 = 1$ or 4
- Understanding whether all the places have already been divided or not. It is good to divide one place at a time in the division algorithm. After mastery, one can combine two places but needs to take care while writing the quotient.
- Not using the skill of estimation to check the answer to the division problem.

It is important to understand how to work with students while teaching the division algorithm so that students can conceptually understand the process and mistakes can be minimized.

In the teaching approach, we initially use concrete objects and connect them with the symbolic form of the division algorithm. Dienes block is one of the Teaching Learning Materials (TLM) which can be used to explain the concept and process behind the division algorithm. Students can easily visualize the process and can understand the algorithm so that they can perform the division algorithm for numbers with any number of digits. Here the teacher needs to provide exposure to the students to work with the TLM for various questions after a whole group discussion.

Consider the example ' $452 \div 4 = ?$ ' that I used to demonstrate my teaching approach. During the initial discussion, students estimate that the answer is a little more than 100. The teacher asks the students to show 452, using Dienes blocks. Subsequently, the place value chart is drawn on the floor, with blocks for Hundreds, Tens and Units in which the students place the appropriate Dienes blocks. Some questions to check conceptual understanding which are asked during the process are –

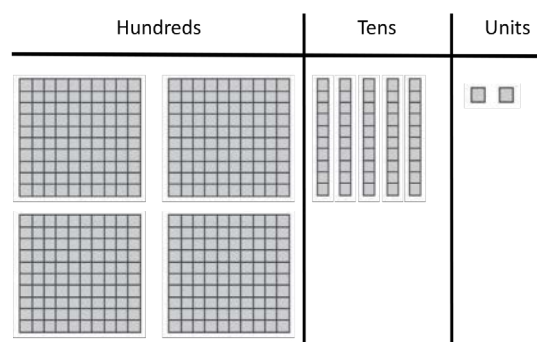


Figure 6

- How many hundreds, tens and units are there in 452,
- Can we write $12 = \underline{\hspace{2cm}}$ units?
- Can we write $52 = 4 \text{ tens} + 12 \text{ units}$?

Then, the teacher uses the concept of equal sharing to discuss the concept of the division algorithm. The teacher draws four circles to show the process of equal sharing and will also write the symbolic form of the problem and connect these with the process.

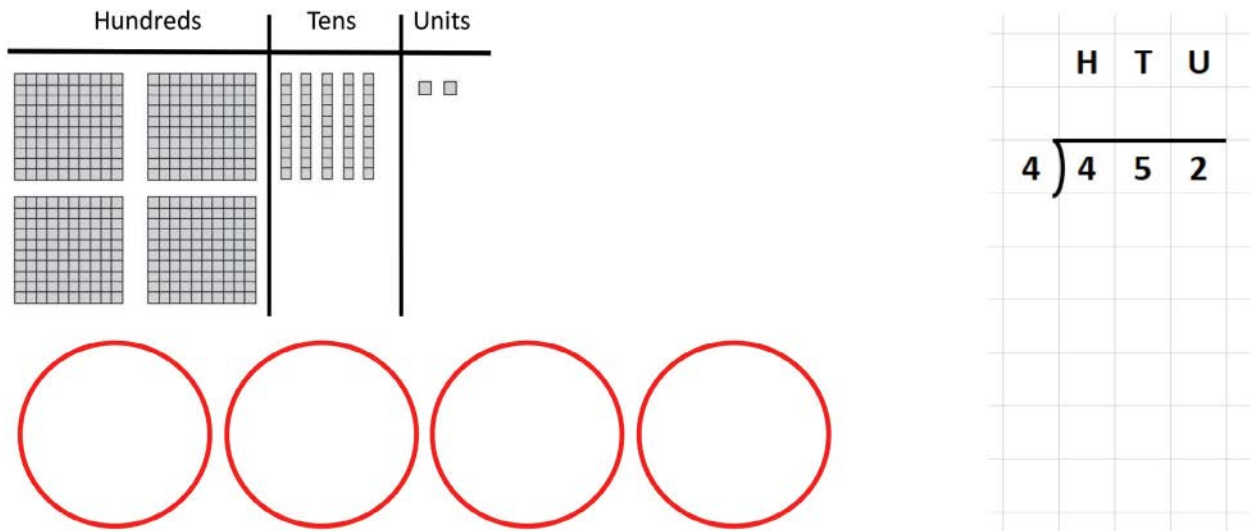


Figure 7

Step 1: First, divide the 4 hundred by 4. That is, divide 4 hundred into 4 groups. We get 1 hundred in each group, which means that 1 is the quotient and the remainder is zero. While explaining the process, we write the symbolic form along.

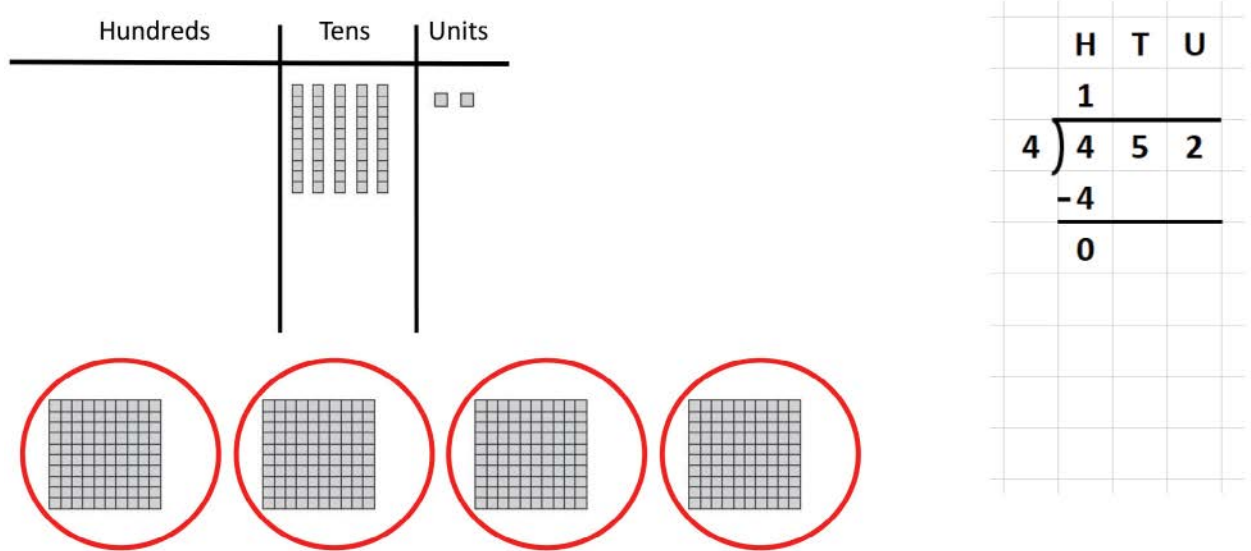


Figure 8

Step 2: Now move to the next place i.e., the tens place. There are 5 tens, and we have to divide by 4 (i.e., into 4 groups). Looking at the blocks, we conclude that we have 1 ten in each group. So we write 1 as the quotient in the Tens place and the remainder is 1 ten.

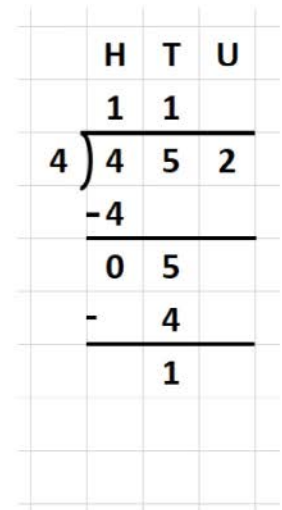
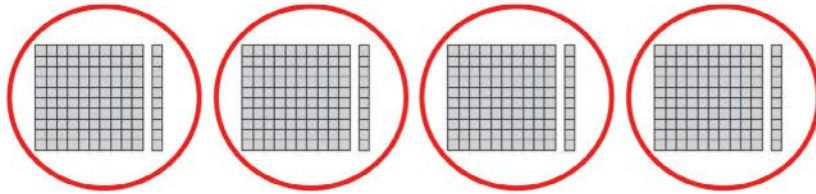
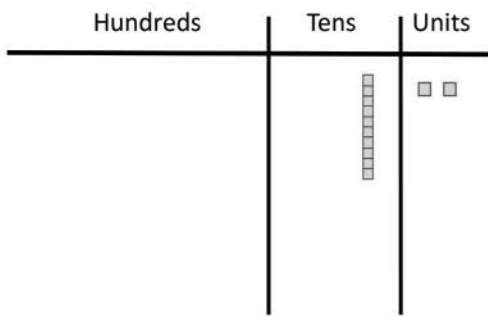


Figure 9

Step 3: Here the remaining ten cannot be divided among the 4 groups but conversion to units is possible. So, adding the 10 units to the 2 units already there, we get 12 units.

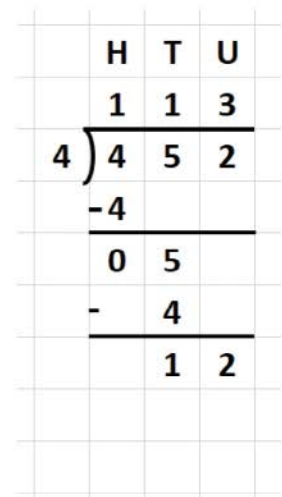
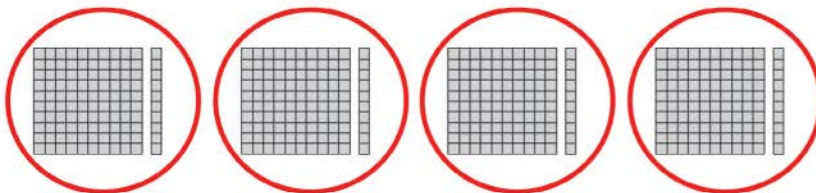
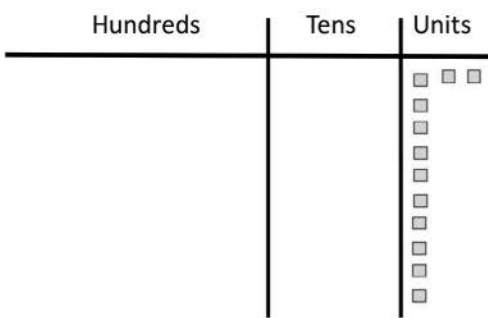


Figure 10

Step 4: Now, we will divide the 12 units among the 4 groups. We will get 3 units in each group i.e.; the quotient has 3 in the units place and there is no remainder.

So, when we divide 452 by 4, we get 1 hundred, 1 tens and 3 units. i.e., $452 \div 4 = 113$

Highlight that in the division algorithm, we repeat the steps Divide, Multiply, Subtract and bring down the next place until all the places have been divided.

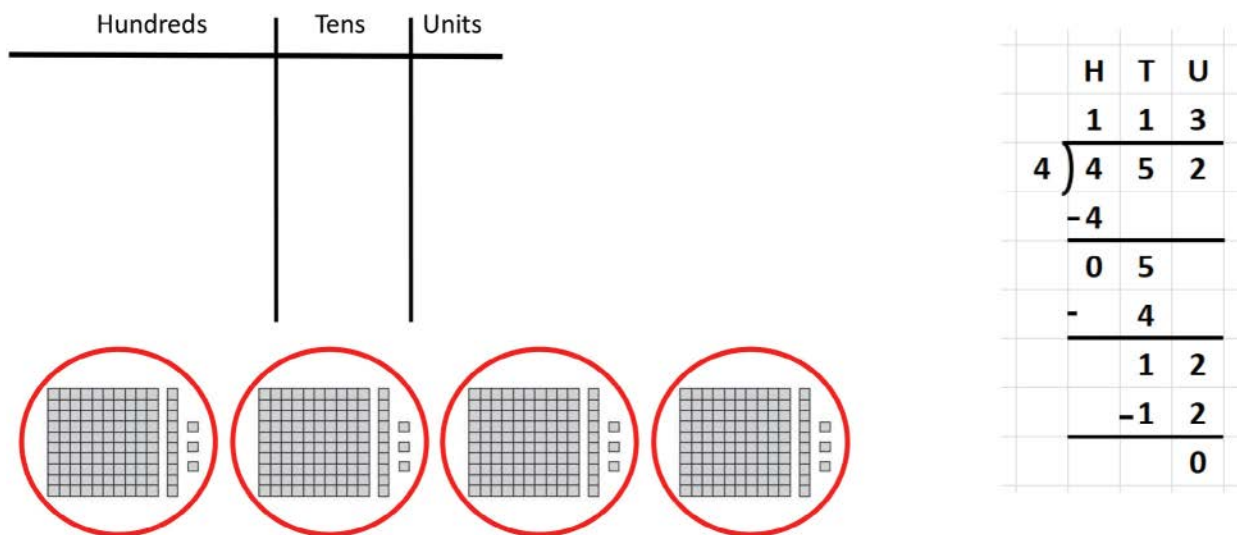


Figure 11

After discussion on the process of $452 \div 4$ and its symbolic form, the teacher can assign a few questions such as $204 \div 2$, $320 \div 4$ etc. in groups of students working with the Dienes blocks.² Check the process and facilitate support to groups who need help. Next, the teacher can assign questions having nonzero remainder and then three-digit by two-digit numbers. In the initial phase provide square grid papers so that the children can write the number as per place value and also can do the division algorithm properly.

In this process, I think we can get a good improvement in the learning of students. Most of the students could then divide whole numbers easily with both conceptual understanding and procedural fluency.

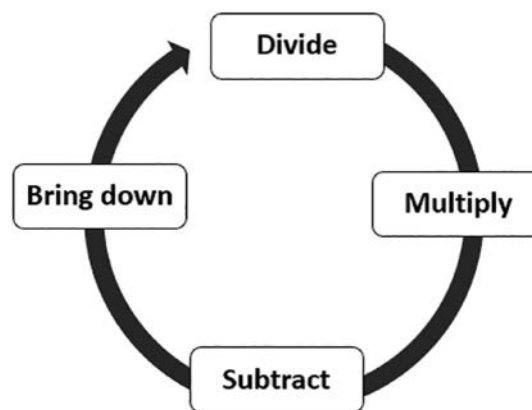


Figure 12



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² Note: The FLU (Flats, Longs, Units) described in the Review section can be made at a very low cost and distributed among the students.

Division with Multi-Digit-Divisors

MATH SPACE

This article is an account of a teacher's work with class 5 post her reading of the article '[Thoughts on the Division Operation](#)', from the July 2015 issue of At Right Angles [1].

Brief Recap: Division is different from addition, subtraction and multiplication in many ways. It is also the most complicated of these four operations mainly because the algorithms for the other three do not require any estimation no matter how large the numbers dealt with are. However, the standard algorithm of long division requires estimation and demands a deeper engagement with a process that navigates many iterations of "if this, then do that".

The current NCERT textbooks do not deal with multi-digit divisors at the preparatory level (till Class 5). Neither do they deal with it later, i.e., at the middle stage (Class 6-8). So, should we teach this at all? Is it needed when calculators are everywhere, even on the 'un'smart phones?

There are two reasons to still teach these:

1. While we may not need to divide by a big number like 365, it is important to know how we can if needed. The process (involving estimation) to divide by a 2-digit number extends for any larger divisor. So, learners should be exposed to division involving 2-digit divisors.
2. A more practical reason, however, is that learners are often expected to divide by a 2-digit number in school. Learners are not allowed to use calculators. Here are some examples:

Keywords: Division algorithm, estimation, reasoning, procedural understanding



7. A milkman sold two of his buffaloes for ₹ 20,000 each. On one he made a gain of 5% and on the other a loss of 10%. Find his overall gain or loss. (Hint: Find CP of each)

10. The population of a place increased to 54,000 in 2003 at a rate of 5% per annum
(i) find the population in 2001.
(ii) what would be its population in 2005?

Chapter 8: Comparing Quantities,
Class 8, NCERT textbook

Ex 8.2, Q7

Ex. 8.3, Q10

Other examples may be found in:-

- Mensuration – Finding the radius of a circle given its circumference, e.g., finding the radius of a circle made by bending a 40cm wire.
- Data Handling – While calculating the mean, e.g., finding the mean of 23 items totalling 4178

Given below is a short narrative from the teacher's experience in doing the problem on Data Handling in her Class 8 mathematics class.



Teacher: We have 23 items with a total weight of 4178 kgs, can anyone estimate the mean weight? You need to explain how you estimated.

Student A: Less than 200 kgs. Because 23×200 is 4600 kg.



Student B: More than 150 kgs. Because 23×100 is 2300 kg, and so the average must be closer to 200 kg than 100 kg.

Teacher: Excellent thinking! I notice that you are using products of multiples of 10s and 100s, to make your estimates. Now, let's see if you can use this to your advantage in dividing by 23. Why don't you round off 23 to the nearest multiple of 10, i.e. 20? Now make an estimate for the first digit of the quotient, i.e., divide 41 by 20.

Student C: We get 2, I can see that we are already going away from the estimated answer!



Teacher: That's observant. So let's check by multiplying 23 by 2, we get 46 and as you said, that's more than 41.

Student B: So, the first digit is 1 and the remainder is $41 - 23$ which is 18. Do we bring down the next digit as we do for single-digit division?

Teacher: Yes, we get 187 and we again try 20×9 which is 180.

Student A: Oh, so we try 23×9 which is 207 and so we then try 23×8 which is 184, that's so close!

Teacher: So, the first two digits of the quotient are 1 and 8 and we get a new remainder of 3, we bring down the next digit 8 and divide 38 by 23.

Student B: I get a quotient of 181 and a remainder of 15. So, my average is about 181 kg, in fact about 181.5 kg because 15 is more than half of 23!



Teacher: Yes, we can continue with decimal division, but for now, this is a great idea of the mean weight. What do you think these 23 items are?

Student D: Well... maybe some sort of animal? Like dolphins? My favourite animals!



Teacher: that's a great suggestion! I know that some motorcycles weigh about 200 kg. Try and find out some other 'items' that weigh 200 kg and let's discuss tomorrow why they would need to find the average weight of these 23 items! Create your reasons! Can we debate the ethics of the scenarios you create?

Hearing this narrative, we at Math Space decided to write down the basic recipe for tackling a 2-digit divisor. Here it is:

1. Round off the divisor to the nearest multiple of 10.
2. Estimate the quotient (or quotient digit) at that step using the estimate.
3. Calculate the product of the quotient digit and the actual divisor.
4. Check
 - For rounding up: if $\text{dividend} - \text{quotient digit} \times \text{divisor} > \text{divisor}$: increase quotient digit by 1 and repeat step 3
 - For rounding down: if $\text{quotient digit} \times \text{divisor} > \text{dividend}$: decrease quotient digit by 1 and repeat step 3
5. Complete the division step with the (modified) quotient.

Since there are so many possible cases, (and later we will see just how many there are), here are some examples. In this article, we will restrict ourselves to 3-digit \div 2-digit. The rest can be generalized as discussed later.

There are multiple possibilities for 3-digit \div 2-digit involving:

- **Rounding up**

Example 1: $672 \div 19$

Round up the divisor to the nearest multiple of 10: 19 rounded off to 20	Estimate the quotient (or quotient digit) at that step using the estimate. $672 \approx 600$, i.e., 6 hundreds $672 \approx 670$, i.e., 67 tens $672 \div 20$ (or $600 \div 20$) $\approx 30 = 3$ tens	
Calculate the product of the quotient digit and the actual divisor:	$3 \text{ tens} \times 19 = 57 \text{ tens}$ (product of quotient digit and actual divisor = 57 tens)	$\begin{array}{r} 30 \\ 19 \overline{)672} \\ \underline{-570} \\ 102 \end{array}$
Check (for rounding up): Here, $\text{dividend} - \text{quotient digit} \times \text{divisor} < \text{divisor}$	$67 \text{ tens} - 57 \text{ tens} = 10 \text{ tens} < 19 \text{ tens}$ \Rightarrow quotient = 3 tens	
Complete the step:	$10 \text{ tens} + 2 \text{ units} = 102$	
Repeat these steps to find the next digit of the quotient.		
Estimating quotient:	$102 \div 20$ (or $10 \div 2$) ≈ 5	$\begin{array}{r} 35 \\ 19 \overline{)672} \\ \underline{-570} \\ 102 \\ \underline{-95} \\ 7 \end{array}$
Calculating:	$5 \times 19 = 95$	
Checking remainder:	$102 - 95 = 7 < 19$ \Rightarrow quotient = 5	
Completing the step:	The quotient is 3 tens + 5 units = 35 and the remainder is 7	

Think about this: How will this change if the divisor is 17 instead of 19, i.e., $672 \div 17$?

Example 2: $867 \div 16$

Round up the divisor to the nearest multiple of 10: 16 rounded up to 20	Estimating quotient: $867 \approx 800$, i.e., 8 hundreds $867 \approx 860$, i.e., 86 tens $867 \div 20$ (or $800 \div 20$) $\approx 40 = 4$ tens	
Calculating:	$4 \text{ tens} \times 16 = 64 \text{ tens}$	$\begin{array}{r} 50 \\ 16 \overline{)867} \\ \underline{-800} \\ 67 \end{array}$
Check for rounding up: Here, dividend – quotient digit \times divisor $>$ divisor, so we increase the quotient digit by 1 and repeat step 3.	$86 \text{ tens} - 64 \text{ tens} = 22 \text{ tens} > 16 \text{ tens}$ \Rightarrow quotient = 4 tens + 1 ten = 5 tens	
Recalculating:	$5 \text{ tens} \times 16 = 80 \text{ tens}$, $86 \text{ tens} - 80 \text{ tens} = 6 \text{ tens} < 16 \text{ tens}$	
Completing the step:	$6 \text{ tens} + 7 \text{ units} = 67$	
Estimating quotient:	$67 \div 20$ (or $6 \div 2$) ≈ 3	$\begin{array}{r} 54 \\ 16 \overline{)867} \\ \underline{-800} \\ 67 \\ \underline{-64} \\ 3 \end{array}$
Now we find the second digit of the quotient.		
Calculating:	$3 \times 16 = 48$	
Checking remainder:	$67 - 48 = 19 > 16$ \Rightarrow quotient = 3 + 1 = 4	
Recalculating:	$4 \times 16 = 64$	
Completing the step:	The quotient is 5 tens + 4 units = 54 and the remainder is 3	

What if the dividend is 863 instead of 867, i.e., $863 \div 16$?

- **Rounding down**

Example 3: $772 \div 31$

Round down the divisor to the nearest multiple of 10: 31 rounded down to 30	Estimating quotient: $772 \approx 700$, i.e., 7 hundreds $772 \approx 770$, i.e., 77 tens $772 \div 30$ (or $700 \div 30$) $\approx 20 = 2$ tens	
Calculating:	$2 \text{ tens} \times 31 = 62 \text{ tens}$	$\begin{array}{r} 20 \\ 31 \overline{)772} \\ \underline{-620} \\ 152 \end{array}$
For rounding down: Here, quotient digit \times divisor $<$ dividend	62 tens is less than 70 tens. \Rightarrow quotient = 2 tens	
Completing the step:	$77 \text{ tens} - 62 \text{ tens} = 15 \text{ tens}$ $15 \text{ tens} + 2 \text{ units} = 152$	
Now we find the second digit of the quotient.		
Estimating quotient:	$152 \div 30$ (or $15 \div 3$) ≈ 5	$\begin{array}{r} 24 \\ 31 \overline{)772} \\ \underline{-620} \\ 152 \\ \underline{-124} \\ 28 \end{array}$
Calculating:	$5 \times 31 = 155$	
Here, quotient digit \times divisor $>$ dividend: so, decrease quotient digit by 1 and repeat step 3	$155 > 153$ \Rightarrow quotient = 5 – 1 = 4	
Recalculating:	$4 \times 31 = 124$ and $124 < 153$	
Completing the step:	The quotient is 2 tens + 4 units = 24 and the remainder is 28.	

How will this change if the dividend is 779 instead of 772, i.e., $779 \div 31$?

Example 4: $805 \div 21$

Rounding: 21 rounded down to 20	Estimating quotient $805 \approx 800$, i.e., 8 hundreds, i.e., 80 tens $805 \div 20$ (or $800 \div 20$) $\approx 40 = 4$ tens	$\begin{array}{r} 30 \\ 21 \overline{)805} \\ \underline{-630} \\ 175 \end{array}$
Calculating:	$4 \text{ tens} \times 21 = 84 \text{ tens}$	
Checking remainder:	$84 \text{ tens} > 80 \text{ tens}$ \Rightarrow quotient = 4 tens – 1 ten = 3 tens	
Recalculating:	$3 \text{ tens} \times 21 = 63 \text{ tens}$	
Completing the step:	$80 \text{ tens} - 63 \text{ tens}$ is 17 tens 17 tens + 5 units is 175	
Now we calculate the next digit of the quotient.		
Estimating quotient:	$175 \div 20$ (or $17 \div 2$) ≈ 8	$\begin{array}{r} 38 \\ 21 \overline{)805} \\ \underline{-630} \\ 175 \\ \underline{-168} \\ 7 \end{array}$
Calculating:	$8 \times 21 = 168$	
Checking remainder:	$168 < 175$ \Rightarrow quotient = 8	
Completing the step:	The quotient is 3 tens + 8 units = 38	

Find the difference if the dividend is 604 instead of 805, i.e., $604 \div 21$?

Examples of divisions with single-digit quotients were discussed in the previous article. The following table summarizes all $2 \times 2 \times (1 + 2) = 12$ possibilities with one numerical case each.

EQ = estimated quotient, FQ = final quotient

	1-step division, 1-digit quotient		2-step division, 2-digit quotient		
		Examples	Step 1	Step 2	Examples
Divisor rounded up	FQ = EQ	$243 \div 37$	FQ = EQ	FQ = EQ	$672 \div 19$
				FQ > EQ	$672 \div 17$
	FQ > EQ	$256 \div 36$	FQ > EQ	FQ = EQ	$863 \div 16$
				FQ > EQ	$867 \div 16$
Divisor rounded down	FQ = EQ	$254 \div 31$	FQ = EQ	FQ = EQ	$779 \div 31$
				FQ < EQ	$772 \div 31$
	FQ < EQ	$256 \div 33$	FQ < EQ	FQ = EQ	$805 \div 21$
				FQ < EQ	$604 \div 21$

Step 1 of the recipe can be modified as follows for bigger divisors:

If the divisor has n -digits, then round it off to the nearest multiple of 10^n .

The rest of the steps remain as they are.

For example, let us consider, $8397 \div 365$

	Rounding:	365 rounded off to 400 $8397 \approx 8000$, i.e., 8 thousand $8397 \approx 8300$, i.e., 83 hundreds	
First digit of quotient	Estimating quotient:	$8397 \div 400$ (or $83 \div 4$) $\approx 20 = 2$ tens	$\begin{array}{r} 20 \\ 365 \overline{)8397} \\ \underline{-730} \\ 109 \end{array}$
	Calculating:	$2 \text{ tens} \times 365 = 730 \text{ tens}$	
	Checking remainder:	$839 \text{ tens} - 730 \text{ tens} = 109 \text{ tens} < 365 \text{ tens}$ \Rightarrow quotient = 2 tens	
	Completing the step:	$839 - 73 \text{ tens} = 109$	
Second digit of quotient	Estimating quotient:	$1097 \div 400$ (or $10 \div 4$) ≈ 2	$\begin{array}{r} 23 \\ 365 \overline{)8397} \\ \underline{-730} \\ 1097 \\ \underline{-1095} \\ 2 \end{array}$
	Calculating:	$2 \times 365 = 730$	
	Checking remainder:	$1097 - 730 = 367 > 365$ \Rightarrow quotient = $2 + 1 = 3$	
	Recalculating:	$3 \times 365 = 1095$	
	Completing the step:	The quotient is 2 tens + 3 units = 23 and the remainder is 2	

We hope that this will help learners navigate the complexity generated by multi-digit divisors, especially the estimation and the nuances it brings along. Notice that as each digit of the quotient is obtained, the recipe focuses on its place value, something that the teacher chose perhaps, to ignore during her discussion with the students.

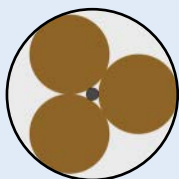
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MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] and their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

Sweet Stuff!

Arjun's mother gave him three gulab jamuns of the same size, each almost perfectly circular, in a circular bowl. Arjun wanted to taste the sugar syrup using a straw. When he placed the straw in the middle, he noticed something unusual. Each gulab jamun seemed to be just touching the others, with each one also just



touching the boundary of the bowl, and the straw just touching each gulab jamun.

If Arjun knows that the radius of the straw is 1 unit,

- Can he find the radius of each of the gulab jamuns?
- Can he find the radius of the bowl?

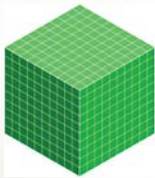



Division with Decimals

**NARAYANA MEHER &
SWATI SIRCAR**

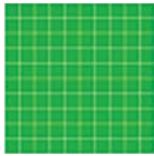


Fractions and decimals are two interconnected crucial concepts with which children face enormous difficulty in middle school (Class 6-8). First, these concepts are difficult to visualise, and second, arithmetic operations on them are usually taught with a lot of abstraction with emphasis given to the rule to be used. Here, we will discuss division with decimal numbers – understanding the operation conceptually– and observe with the help of manipulatives how the rule emerges.

Before we move to division with decimals, we need to understand the meaning of the division operation. It has two meanings, one is *equal sharing*, i.e., $12 \div 3$ means 12 things distributed equally among 3 groups (i.e., how many does each of the 3 groups get), and another is *equal grouping (measure)*, i.e., 12 things distributed in a way such that each gets 3 (i.e., how many groups get 3 things each).

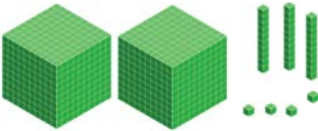
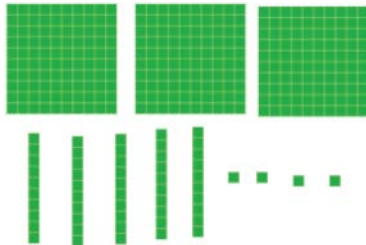
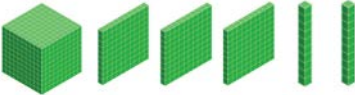
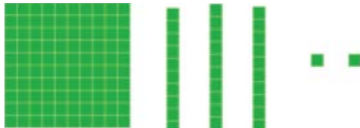
Both 3D and 2D manipulatives made with cardboard/paper may be used for modelling decimals. Whichever we use, there is a need for consistency throughout the representations and the problem-solving.

3D model of decimals	The whole, the big cube or 1. 	The whole can be split into 10 equal plates, each being $\frac{1}{10}$ or 0.1 of the whole. 	Each plate can be further split into 10 equal rods, each being $\frac{1}{100}$ or 0.01 of the whole. 	Each rod can be further split into 10 equal small cubes, each being $\frac{1}{1000}$ or 0.001 of the whole. 
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Keywords: TLMs, Procedural Understanding, Operations on Decimals

2D model of decimals	The whole, the flat or 1. 	The whole can be split into 10 equal longs, each being $\frac{1}{10}$ or 0.1 of the whole. 	Each long can be further split into 10 equal small squares, each being $\frac{1}{100}$ or 0.01 of the whole. 
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Representing decimals using these models

2.034 using the 3D model 	3.54 using 2D model 
1.32 using 3D model 	1.32 using 2D model 

We will try to perform division with decimals using either the 3D or the 2D model, as needed. One can use Mathigon Polypad (<https://mathigon.org/polypad>) for virtual models while the physical ones can be made with cardboard/paper. 2D models can be extended to 4 places after the decimal point, i.e., 0.0001 using a centimeter graph paper with 10cm × 10cm square as a whole.

Prerequisites for division with decimals

- *Understanding of decimals, especially expressing decimals as fractions*

For example, $0.25 = \frac{25}{100} = \frac{1}{4}$

If one flat is a whole or 1 which consists of 100 small squares, each 0.01, (Figure 1).

0.25 is 25 such small squares, i.e., 25×0.01 . 4 such 0.25 make the whole as shown in Figure 2. So, 0.25 is $\frac{1}{4}$ of a whole or flat.

- *Multiplication with decimals, specifically products of decimals and powers of ten*

For example, $0.34 \times 0.002 = \frac{34}{100} \times \frac{2}{1000} = \frac{34 \times 2}{100 \times 1000} = \frac{68}{100000} = 0.00068$

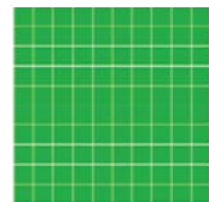


Figure 1

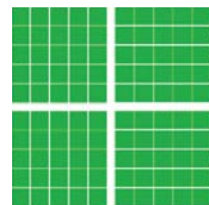


Figure 2

- *Division with fractions, specifically, division by a fraction being the same as multiplication by the reciprocal of the fraction*

For example, $\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times \frac{5}{2}$ (multiplied by the reciprocal of $\frac{2}{5}$) $= \frac{15}{8} = 1\frac{7}{8}$

Let us explore four possible division situations with decimals as given below.

Types/Situations

1. Natural number \div Natural number = Decimal
2. Decimal \div Natural number = Decimal
3. Natural number \div Decimal
 - a. = Natural number
 - b. = Decimal
4. Decimal \div Decimal
 - a. = Natural number
 - b. = Decimal

So, both dividends and divisors can be natural numbers or decimals. As we proceed further, we will realize that the nature of the dividend is less important than that of the divisor. (The same thing is observed in division with fractions.)

Natural Number \div Natural Number = Decimal

In this case both the dividend and divisor are whole numbers.

If the divisor is a natural number, then it can be considered as **the number of groups among whom the dividend is to be divided equally**, i.e., using the **equal sharing** meaning of division. For example, in $12 \div 40$, 12 is divided equally among 40 groups. Let us see how this can be done through the 3D model.

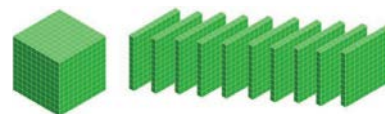


Figure 3

Now 12 cubes cannot be directly divided by 40. So, we convert each cube to 10 plates (Figure 3).

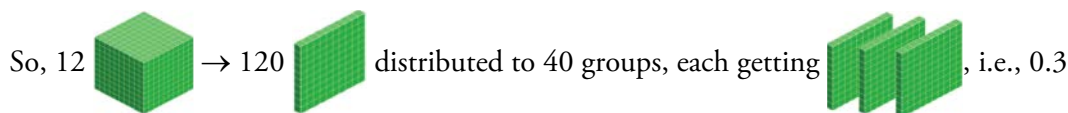


Figure 4

$\therefore 12 \div 40 = 0.3$ (Figure 4)

If we do the same division using the 2D model, 12 flats cannot be distributed among 40 groups. So, we convert each flat to 10 longs (Figure 5).

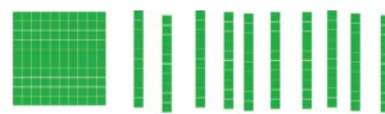


Figure 5

So, 12 \rightarrow 120 distributed among 40 groups, each group got (0.3), i.e., $12 \div 40 = 0.3$ (Figure 6).

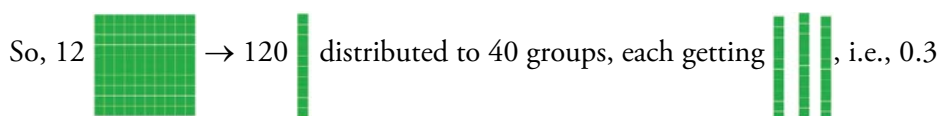


Figure 6

We get the same quotient using either model.

Decimal ÷ Natural Number = Decimal

Let us take the example $0.24 \div 5$. Again, since the divisor is a natural number, we can use the [equal sharing](#) meaning of division. Therefore, in this case, 0.24 has to be divided equally among 5 groups.

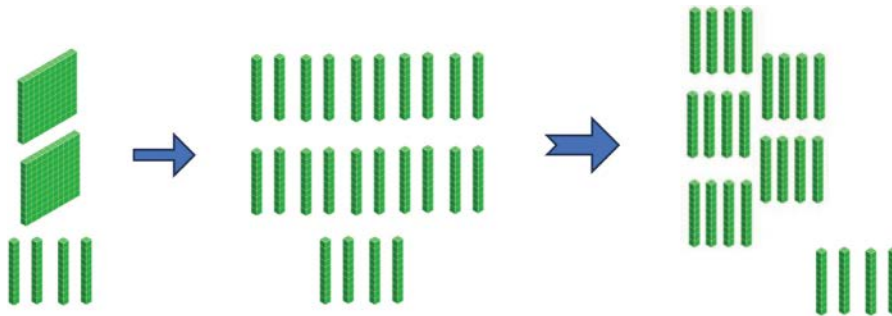


Figure 7

In the first round, 2 plates are converted into 20 rods, equally distributed among 5 groups, each group gets 4 rods (0.04), and 4 rods are left over since they can't be distributed to 5 groups (Figure 7).

In the second round, 4 rods are converted into 40 small cubes, and equally distributed among 5 groups, each group gets 8 small cubes (0.008) (Figure 8).



Figure 8

So, combining both rounds of distribution, each group gets $0.04 + 0.008 = 0.048$, i.e., $0.24 \div 5 = 0.048$ (Figure 9).

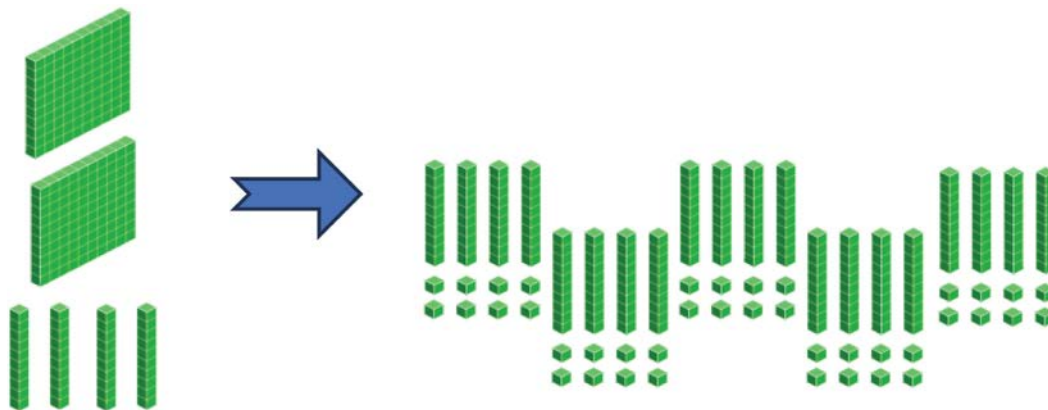


Figure 9

In an alternative way, we can convert 0.24, i.e., 2 plates and 4 rods into 240 small cubes (0.001) and distribute them among 5 groups, so that each group gets 48 small cubes, i.e., 0.048. So, $0.24 \div 5 = 0.048$ in either way.

Natural Number ÷ Decimal = Whole number

However, if the divisor is not a natural number, then it can't represent a quantity like the number of groups. So, if the divisor is a decimal number, we can't use the [equal sharing meaning](#). Therefore, it makes

more sense to use the **equal grouping (measure) meaning**, i.e., the divisor amount is given to each group and the number of groups is the quotient.

So, for $12 \div 0.3$, each group gets 0.3 or 3 plates (0.1). 12 cubes can be converted to 120 plates. Since each group gets 3 plates, 120 plates can be distributed among $120 \div 3 = 40$ groups (Figure 10). This works fine if the quotient is a whole number.

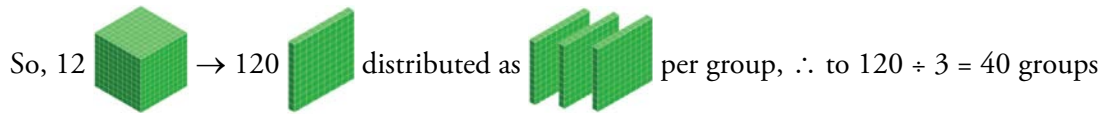


Figure 10

Natural Number \div Decimal = Decimal

But when the quotient is not a whole number, then this meaning of division does not suffice. For example, it does not help with $11 \div 0.4$.

11 cubes can be converted into 110 plates. If we divide 110 plates among certain groups where each group gets 4 plates, i.e., $110 \div 4$ (Figure 11).



Figure 11

Then 108 (out of 110) plates can be equally distributed among 27 groups. But 2 plates, i.e., 0.2, is left and cannot be distributed further since it is too little ($0.2 < 0.4$). So, **equal grouping** fails to make sense in this case using manipulatives.

Decimal \div Decimal = Natural Number

Let us consider $0.42 \div 0.14$. So, 0.42 (4 plates and 2 rods) is divided into certain groups such that each group gets 0.14 (1 plate and 4 rods). Now, 4 plates and 2 rods can be converted into 3 plates and 12 rods. So, these can be distributed among 3 groups, each getting 1 plate and 4 rods, i.e., 0.14 (Figure 12).

$$\therefore 0.42 \div 0.14 = 3$$

So, in this case, **equal grouping** makes sense since the quotient is a natural number.

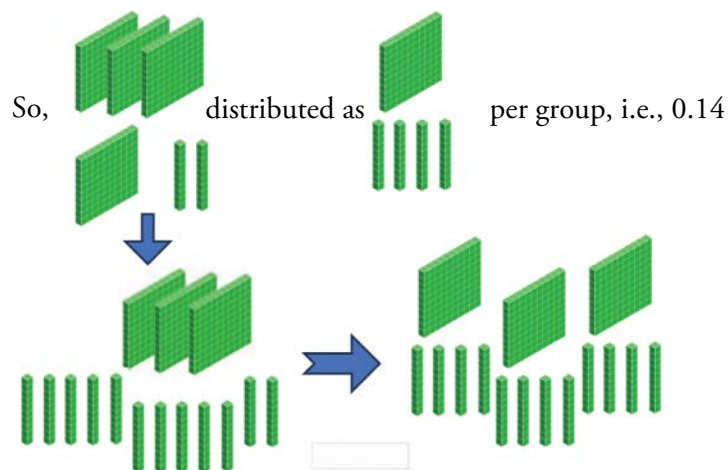


Figure 12

Decimal ÷ Decimal = Decimal Number

Let us now consider $0.42 \div 1.4$. If we try to solve this by using manipulatives, let us see what issues we may face. For this case let us use 2D modeling instead of 3D.

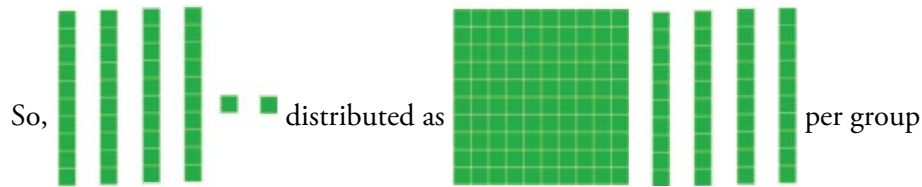


Figure 13

Clearly, the amount to be divided, i.e., the dividend 0.42, is less than how much each group must get, i.e., the divisor 1.4. Therefore, it is impossible to use the **equal grouping meaning** to make sense of this division using manipulatives.

Let us consider $2.03 \div 0.5$. In this case, dividend 2.03 is larger than the divisor 0.5. But still, the division remains incomplete (Figure 14) because 0.03 cannot be distributed since it is too small ($0.03 < 1.4$). And so **equal grouping** fails again.

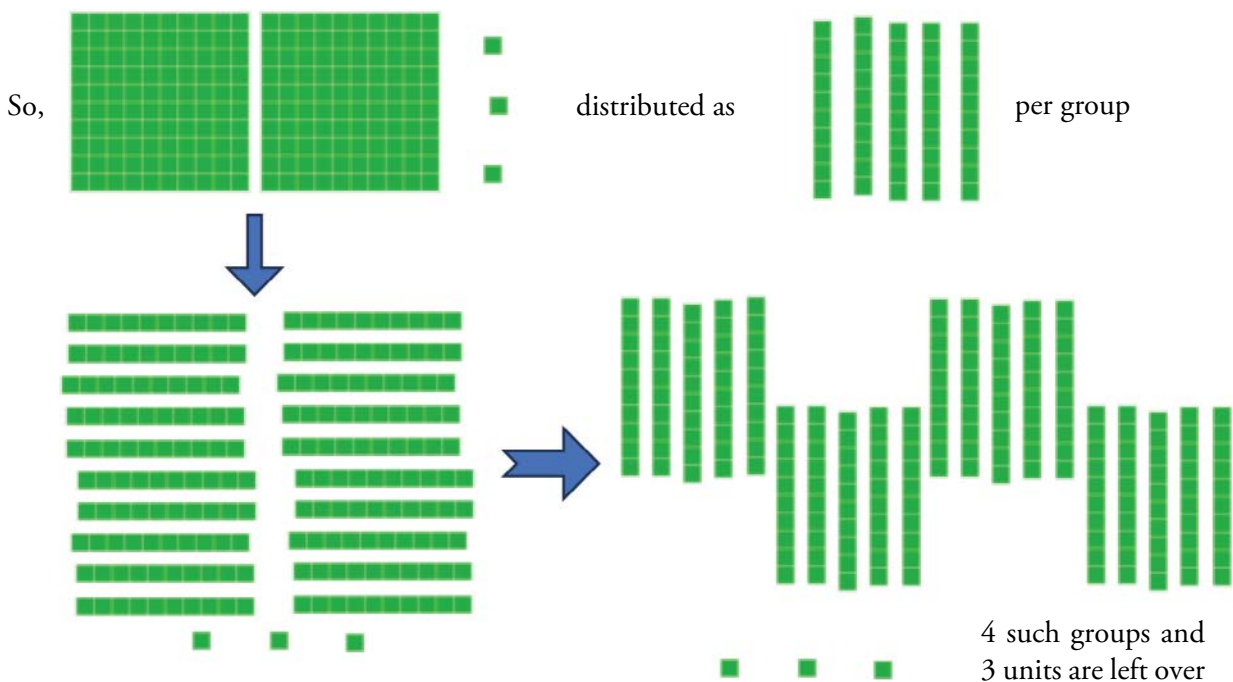


Figure 14

Note that this is not a limitation of manipulatives. For example, $0.000042 \div 0.000014$ cannot be demonstrated with manipulatives but can be explained with **equal grouping** if we can imagine tiny pieces representing 0.00001 and 0.000001. We got this clarity only when we explored.

So, whenever the **divisor** is a natural number, **equal sharing** can be used to make sense of the division. Similarly, whenever the **quotient** is a natural number, **equal grouping** helps in understanding the corresponding division. But if neither the divisor nor the quotient is a natural number, then how

can we make sense of such a division? In particular, that happens in the following two cases as illustrated above:

1. natural number \div decimal = decimal
2. decimal \div decimal = decimal

One possibility is to use division of fractions, since decimals can be converted to fractions and the chocolate plate model* adequately illustrates all possible situations for division with fractions (and natural numbers).

***Chocolate Plate model:** For $p \div q$, let p be the amount of chocolate to be distributed among q plate(s). The quotient is the amount of chocolate on one plate. Both p and q can be any natural number or any fraction, i.e., unit, proper or even improper.

Let us consider the cases where both meanings of division failed, i.e., (i) $11 \div 0.4$, (ii) $0.42 \div 1.4$ and (iii) $2.03 \div 0.5$.

$$(i) \quad 11 \div 0.4 = 11 \div \frac{4}{10} = 11 \times \frac{10}{4} = \frac{11 \times 10}{4} = 110 \div 4$$

Note that we had arrived at the same thing with **equal grouping**, but there the quotient represented the number of groups and so, it had to be a whole number. But we have no such restriction here since we are not using that meaning. Now we are free to use the **equal sharing meaning** for $110 \div 4$.

Also note that $110 \div 4 = (11 \times 10) \div (0.4 \times 10)$, i.e., both the dividend and divisor are multiplied by 10 to shift the decimal points and make the divisor a natural number.

$$(ii) \quad 0.42 \div 1.4 = \frac{42}{100} \div \frac{14}{10} = \frac{42}{100} \times \frac{10}{14} = \frac{42}{140} = 4.2 \div 14$$

Note that if we convert just the divisor to a fraction, then also we end up with the same thing since

$$0.42 \div \frac{14}{10} = 0.42 \times \frac{10}{14} = \frac{0.42 \times 10}{140} = 4.2 \div 14 = (0.42 \times 10) \div (1.4 \times 10)$$

In both cases, we have again effectively multiplied both dividend and divisor by the same power of 10. This was done to shift the corresponding decimal points. With such shifts, the divisor becomes a natural number.

$$(iii) \quad 2.03 \div 0.5 = 2.03 \div \frac{5}{10} = 2.03 \times \frac{10}{5} = \frac{2.03 \times 10}{5} = 20.3 \div 5 = (2.03 \times 10) \div (0.5 \times 10), \text{ i.e., both the}$$

dividend and divisor are again multiplied by the same number, shifting the decimal points, and making the divisor a natural number.

This is exactly what is done as the usual procedure, but without any explanation.

So, for a general process that works for all types of division with decimal divisors, it makes more sense (i) to convert the divisor, to a fraction, whose denominator is a power of 10, and (ii) then use division by

a fraction. As we convert this 'division by a fraction' to a 'multiplication by the reciprocal of the fraction', the dividend gets multiplied by a power of 10. This power of 10 is the denominator of the decimal divisor. And this product is the new dividend. The new divisor is the numerator of the original decimal divisor, a natural number.

Decimal divisor (DD) = natural number/power of 10 = $N/10^m$

Original dividend (OD) \div DD = $OD \div N/10^m = (OD \times 10^m) \div N$

For example: Let us consider $3.006 \div 0.15$, here DD = $0.15 = 15/100$, i.e. $N = 15$ and $m = 2$, while OD = 3.006. So, $3.06 \div 0.15 = 3.006 \div 15/10^2 = (3.006 \times 10^2) \div 15 = 300.6 \div 15$.

Note that it is enough to make the divisor a natural number, and then we can use [equal sharing](#). It does not matter if the dividend remains a decimal or not.

Throughout this discussion we have deliberately avoided recurring decimals since they do not appear till the secondary stage according to the current syllabus. Therefore, we felt that recurring decimals are not relevant enough. However, the process remains the same even when the quotient is a recurring decimal.



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Solution to Lights Off! [Page 11]

1. Suppose the house is a 3×3 square with 9 rooms. If all the lights in all the rooms are initially in the 'on' position can you turn all the lights off? What is the minimum number of moves required to achieve this?
2. Can you **generalise** this for an $n \times n$ square?
3. Can you **extend** the problem to the case where house is the shape of an $m \times n$ rectangle?
4. Can you **extend** the problem to a $2 \times 2 \times 2$ cube?
5. What other kinds of extensions and generalisations can you think of? How would you approach solving these problems?

Send in your solutions and/or any problems you created to AtRightAngles.editor@apu.edu.in

Worksheet on division with decimals

1. Try the following:

- a. $13 \div 4 =$
- b. $7 \div 8 =$
- c. $3.4 \div 5 =$
- d. $0.9 \div 20 =$

2. What do you see?

- a. $12 \div 3 =$
- b. $12 \div 0.3 = 12 \div \frac{3}{\square} = 12 \times \frac{\square}{3} = \frac{\square}{\square} = \text{_____} \div 3$
- c. $12 \div 0.03 = 12 \div \frac{3}{\square} = 12 \times \frac{\square}{3} = \frac{\square}{\square} = \text{_____} \div \text{_____}$
- d. $12 \div 0.003 = 12 \div \frac{3}{\square} = 12 \times \frac{\square}{\square} = \frac{\square}{\square} = \text{_____} \div \text{_____}$

Do you observe a pattern?

Each time, the divisor is written as a _____, whose _____ is a power of _____.

The given division = given dividend \times denominator of given divisor \div numerator of given divisor.

Note that, this is equivalent to shifting the decimal point of both dividend and divisor to the right till the divisor becomes a natural number.

3. So, fill in the blanks with natural numbers and find the quotients.

- a. $26 \div 0.5 = \text{_____} \div \text{_____} =$
- b. $7 \div 0.08 = \text{_____} \div \text{_____} =$
- c. $3 \div 0.12 = \text{_____} \div \text{_____} =$

4. Extending the same process,

- a. $1.05 \div 7 =$
- b. $1.05 \div 0.7 = 1.05 \div \frac{7}{\square} = 1.05 \times \frac{\square}{7} = \frac{\square}{7} = \text{_____} \div 7$
- c. $1.05 \div 0.07 = 1.05 \div \frac{7}{\square} = 1.05 \times \frac{\square}{7} = \frac{\square}{7} = \text{_____} \div \text{_____}$
- d. $1.05 \div 0.007 = 1.05 \div \frac{7}{\square} = 1.05 \times \frac{\square}{7} = \frac{\square}{7} = \text{_____} \div \text{_____}$

Again, the given division = dividend \times denominator of given divisor \div numerator of given divisor.

5. So, complete the following divisions:

- a. $1.7 \div 0.02 = \text{_____} \div \text{_____} =$
- b. $0.003 \div 0.05 = \text{_____} \div \text{_____} =$
- c. $0.36 \div 0.9 = \text{_____} \div \text{_____} =$

Divisibility Rules for 7

JITENDRA VERMA

Divisibility rules are one of the important topics of study in school mathematics, especially in upper primary classes. They enable us to quickly identify if one number is divisible by another. We know various methods for checking the divisibility of a number by 2, 3, 4, 5, 6, 8, 9, 10 etc. It is also clear that checking of divisibility of the given number by some numbers is quite easy, while for some numbers is a bit complicated.

Divisibility by 7 is a challenging one, with many attempts made to simplify the rule. [Chika's divisibility rule for 7](#) is a recent one among them. Here, we shall discuss three different divisibility methods for 7, using existing methods which add new dimensions to the concept.

Method 1: Doubling the unit digit

Take the given number	Remove the unit digit and write the truncated number	Double the unit digit which was removed	Subtract the doubled digit from the truncated number	If the difference is either 0 or a multiple of 7, then the original number is divisible by 7. (Repeat if necessary)
532	53	$2 \times 2 = 4$	$53 - 4 = 49$	49 is divisible by 7 so 532 is also divisible by 7
427	42	$2 \times 7 = 14$	$42 - 14 = 28$	28 is divisible by 7 so 427 is also divisible by 7
29792	2979	$2 \times 2 = 4$	$2979 - 4 = 2975$	Repeat for 2975
2975	297	$2 \times 5 = 10$	$297 - 10 = 287$	Repeat for 287
287	28	$2 \times 7 = 14$	$28 - 14 = 14$	14 is divisible by 7 so 29792 is also divisible by 7
Try 2308012 now				

With the above examples, we understand that this method is useful for checking divisibility by 7 without performing long division for a 3-digit number, but is quite lengthy for 4 or more-digit numbers.

Keywords: Factors, divisibility, checks, rules, justification

Justification of the rule

Suppose $N = 1000 a_3 + 100 a_2 + 10 a_1 + a_0$

(Where a_0, a_1, a_2, a_3 are the digits of the 4-digit number N)

According to the rule, we write the truncated version (say N_T) without the unit digit of N and then take away (subtract) from N_T twice the unit digit to get a new number (say M).

$N_T = 100 a_3 + 10 a_2 + a_1$ (Note the change in the place values after the number is truncated)

$M = N_T - 2a_0 = 100 a_3 + 10 a_2 + a_1 - 2a_0$

Our rule says that if M is a multiple of 7, then N is also a multiple of 7.

Assume that M is a multiple of 7, i.e. $M = 7k$ for some whole number k .

Then, $M = 7k = 100 a_3 + 10 a_2 + a_1 - 2a_0$ or $100 a_3 + 10 a_2 + a_1 = 7k + 2a_0$

Substituting this in N , we get

$N = 1000 a_3 + 100 a_2 + 10 a_1 + a_0$

$N = (1000 a_3 + 100 a_2 + 10 a_1) + a_0 = 10(100 a_3 + 10 a_2 + a_1) + a_0$
 $= 10(7k + 2a_0) + a_0 = 70k + 21a_0 = 7(10k + 3a_0)$

So, if M is a multiple of 7, then so is N .

This can easily be generalized to any number of digits.

Method 2: Multiplying the unit digit by 5

Take the given number	Remove the unit digit and write the truncated number	Multiply the unit digit by 5	Add the result to the truncated number	If the sum is either 0 or a multiple of 7, then the original number is divisible by 7 (Repeat if necessary)
378	37	$8 \times 5 = 40$	$37 + 40 = 77$	77 is divisible by 7 so 378 is also divisible by 7
2464	246	$5 \times 4 = 20$	$246 + 20 = 266$	Repeat for 266
266	26	$5 \times 6 = 30$	$26 + 30 = 56$	56 is a multiple of 7, So 266 and 2464 are divisible by 7
29792	2979	$2 \times 5 = 10$	$2979 + 10 = 2989$	Repeat for 2989
2989	298	$9 \times 5 = 45$	$298 + 45 = 343$	Repeat for 343
343	34	$3 \times 5 = 15$	$34 + 15 = 49$	49 is a multiple of 7 so 343, 2989 and 29792 are divisible by 7
Try 2308012 now				

One may provide a justification, which is very similar to the previous one as follows.

Suppose $N = 1000 a_3 + 100 a_2 + 10 a_1 + a_0$

(Where a_0, a_1, a_2, a_3 are the digits of the 4-digit number N)

According to the rule, we write the truncated version (say N_T) of N and add five times the unit digit to get a new number (say M).

$$N_T = 100 a_3 + 10 a_2 + a_1$$

$$M = N_T + 5a_0 = 100 a_3 + 10 a_2 + a_1 + 5a_0$$

Our rule says that if M is a multiple of 7, then N is also a multiple of 7.

Assume that M is a multiple of 7, i.e. $M = 7k$ for some whole number k .

$$\text{Then, } M = 7k = 100 a_3 + 10 a_2 + a_1 + 5a_0 \text{ or } 100 a_3 + 10 a_2 + a_1 = 7k - 5a_0$$

Substituting this in N , we get

$$N = 1000 a_3 + 100 a_2 + 10 a_1 + a_0$$

$$\begin{aligned} N &= (1000 a_3 + 100 a_2 + 10 a_1) + a_0 = 10(100 a_3 + 10 a_2 + a_1) + a_0 \\ &= 10(7k - 5a_0) + a_0 = 70 k - 49 a_0 = 7(10k - 7 a_0) \end{aligned}$$

So, if M is a multiple of 7, then so is N .

This can easily be generalized to any number of digits.

Method 3: Grouping of digits (Rule – 1-3-2)

Take the Number	Make groups of three digits starting from the unit digit	Multiply the right-most digit by 1, the next by 3 and the left-most by 2 in each group	Add all odd-numbered groups	Add all even-numbered groups	Difference $ c-d $
	a	b	c	d	e
$N_1 = 672$	672	$6 \times 2 + 7 \times 3 + 2 \times 1 = 35$	35	0	35
Result	$ c-d = 35$ is divisible by 7. So, the number N_1 is also divisible by 7.				
$N_2 = 4704$	004 704	$4 \times 1 = 4$ $7 \times 2 + 0 \times 3 + 4 \times 1 = 18$	18	4	$ 18 - 4 = 14$
Result	$ c-d = 14$ is divisible by 7. So, the number N_2 is also divisible by 7.				
$N_3 = 32921$	032 921	$3 \times 3 + 2 \times 1 = 11$ $9 \times 2 + 2 \times 3 + 1 \times 1 = 25$	25	11	$ 25 - 11 = 14$
Result	$ c-d = 14$ is divisible by 7. So, the number N_3 is also divisible by 7.				
$N_4 = 197526$	197 526	$1 \times 2 + 9 \times 3 + 7 \times 1 = 36$ $5 \times 2 + 2 \times 3 + 6 \times 1 = 22$	22	36	$ 22 - 6 = 14$
Result	$ c-d = 14$ is divisible by 7. So, the number N_4 is also divisible by 7.				
$N_5 = 164953525268$	164 953 525 268	$1 \times 2 + 6 \times 3 + 4 \times 1 = 24$ $9 \times 2 + 5 \times 3 + 3 \times 1 = 36$ $5 \times 2 + 2 \times 3 + 5 \times 1 = 21$ $2 \times 2 + 6 \times 3 + 8 \times 1 = 30$	$30+36 = 66$	$21+24 = 45$	$ 66 - 45 = 21$
Result	21 is divisible by 7. So, the number N_5 is divisible by 7.				

This is yet another way of checking for divisibility by 7. Let's illustrate it step by step.

1. Starting from the unit place of the number, make groups of three digits. The last group will contain the remaining digits.
2. In each group, multiply the right-most digit by 1, the next by 3 and the left-most by 2.
3. Add all the products obtained in each group.
4. Find the sums of the odd and even-numbered groups.
5. If the difference of these two sums is divisible by 7 or is 0, then the original number will be divisible by 7.

Justification of the rule

Suppose $N = 100000 a_5 + 10000 a_4 + 1000 a_3 + 100 a_2 + 10 a_1 + a_0$

(Where $a_0, a_1, a_2, a_3, a_4, a_5$, are the digits of the 6-digit number N)

$$S_1 = a_0 \times 1 + a_1 \times 3 + a_2 \times 2 \qquad S_2 = a_3 \times 1 + a_4 \times 3 + a_5 \times 2$$

$$M = S_1 - S_2$$

Our rule says that if M is a multiple of 7, then N is also a multiple of 7.

Assume that M is a multiple of 7, i.e. $M = 7k$ for some whole number k .

$$\begin{aligned} 7k &= (a_0 \times 1 + a_1 \times 3 + a_2 \times 2) - (a_3 \times 1 + a_4 \times 3 + a_5 \times 2) \\ &= (-2a_5 - 3a_4 - a_3 + 2a_2 + 3a_1 + a_0) \end{aligned}$$

$$N = (100002a_5 - 2a_5) + (10003a_4 - 3a_4) + (1001a_3 - a_3) + (98a_2 + 2a_2) + (7a_1 + 3a_1) + a_0$$

$$N = 7(14286a_5 + 1428a_4 + 143a_3 + 14a_2 + a_1) + (-2a_5 - 3a_4 - a_3 + 2a_2 + 3a_1 + a_0)$$

$$N = 7(14286a_5 + 1428a_4 + 143a_3 + 14a_2 + a_1) + 7k$$

So, if M is a multiple of 7, then so is N .

This can easily be generalized to any number of digits.

Note: Rule – 1-3-2 can be used for any number with 2 or more digits. It can help us to find the divisibility of any number by 7 easily and quickly as well.

Comparison

Method	Operations needed	Remark
Doubling the unit digit	$\times, -$	Useful for 2 or 3-digit numbers.
Unit digit is multiplied by 5	$\times, +$	Useful for 2 or 3-digit numbers.
Rule 132	$\times, +, -, \text{grouping}$	Useful for more than 3-digit numbers.

Explorations such as this help teachers plan lessons in which students *develop capacities for problem-solving, logical reasoning, and computational thinking. Students become comfortable in working with abstractions and other core techniques of Mathematics and Computational Thinking, such as the mathematical modelling of phenomena and the development of algorithms to solve problems.* (NCF-SE 2023).

If the teaching of divisibility rules stops at practising number skills, then we are severely limiting the potential of such a rich topic. Asking why the rule works, trying to generalise it, comparing different rules and then trying to make their own rules will not just develop mathematical minds but also impart an understanding of the joy and beauty of the subject.

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Math is a cake walk!

At Right Angles November 2023 Challenge

We divided a delicious chocolate cake into 12 pieces and served each piece in a half plate!

Here is your challenge!

How many math questions can you make from this situation?

Send in your questions to AtRiA.editor@apu.edu.in



Response from reader

Rohini Khaparde

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School Of Scholars, Akola.

Affiliation No. 1130166

1. How many plates will be needed to serve two-thirds of the cake?
2. If out of 12 pieces only 8 are to be served, then what is the ratio of the number of plates needed to the total number of plates needed to serve the full cake?

Response from reader

Astik Yadav

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School of Scholars Hudkeshwar Nagpur

1. If the radius of the original circular cake before cutting is 'r' and the cake is cut into 12 equal pieces, each served on a half plate with a radius of 'p', express the ratio of the area of one cake piece to the area of one half plate in terms of 'r' and 'p'.
2. If someone ate one-third of the cake, what fraction of cake is remaining?
3. What percentage of the cake is on each plate if you consider the whole cake as 100%?
4. If someone eats 3 pieces of cake, what fraction of the whole cake has he consumed?
5. If we want to share the cake equally among 4 people, how many pieces would each person get?

Problem Solving – Your Own Way!

SANDEEP DIWAKAR

For some time now, I have had many opportunities to work with teachers and children at both primary and middle school levels. I developed a habit of asking some puzzles or questions related to daily life during our mathematics sessions to make them more enjoyable. When someone came up with a solution, I would always try to understand the process and the thinking behind their processes.

Here is one such question –

“There are some rabbits and chickens in a room. Somebody asked the watchman about the number of chickens and the rabbits. The watchman said that he does not know how many rabbits and chickens there are, but he does know that there are 100 heads and 250 legs in total. Can you find out how many rabbits and how many chickens are in that room?”¹

Whenever this question was posed, whether to adults, middle school children, or teachers, the majority would guess and solve it using the “trial and error method.” They would estimate, for example, that there are 20 chickens and 80 rabbits, giving a total of 100 heads. Then this would give 240 legs, so their estimate needed to be corrected. They would then change the number of rabbits and chickens and again calculate the number of legs and try to get 100 heads and 250 legs.

When challenged with whether there could be more than one way to solve this question, some adults, middle school children, and teachers familiar with algebra would quickly formulate equations to solve the question i.e., $x + y = 100$ and $2x + 4y = 250$ (where x is the number of chickens and y is the number of rabbits).

¹ I read or heard this question somewhere a long time ago.

Keywords: Problem-solving, reasoning, individuals

They would then simplify these equations using methods such as substitution, elimination, or cross multiplication, reducing them to an equation with only one variable. For instance, assuming that the number of rabbits is x , therefore the number of chickens would be $100 - x$. This leads to the equation $4x + 2(100 - x) = 250$, which helps in finding the value of x and calculating the number of rabbits and chickens.

Working with more than 50 groups on this question, I observed that almost all of them resorted to either trial and error or algebraic equations for a solution. Middle school mathematics teachers or those familiar with algebra immediately chose the more formal approach. In contrast, primary teachers and adults who avoid algebra, predominantly used the trial and error method.

When asked about the appropriate level of children who should be asked this question, all mathematics teachers in one voice agreed that this was a difficult question and that unless the children knew algebra, this question should not be asked. They suggested that this question would be fine for children in grade 6 or higher.

This question was also presented to some groups of children at the primary level. Many children could not solve it, saying that this question was different as it did not ask them to add, subtract, multiply or divide directly. Some children tried

to solve it by drawing pictures in different ways but then stopped saying that the question was difficult.

In the course of this work, I found two or three children whose answers were correct and methods unique. When I asked one of them about his approach, he explained in his own language – “Look sir, in this question, there are rabbits and chickens. Rabbits have four legs and chickens have only two legs. Since the total number of heads is 100, I know there are 100 animals. Let’s first give two legs to each, which will take up 200 out of the 250 legs and we will have 50 legs left. Now you cannot give one leg to any animal as no animal has three legs. So you have to give feet in pairs of two. We have 50 legs left, now if we give two legs to each, we will be able to give 50 legs to 25 animals only. In this way, the 25 animals that will get these legs will have four legs, so that these 25 must be rabbits and the remaining 75 animals will be chickens”.

Of course, adults, middle school children, and teachers follow the methods given in textbooks to solve this question, but primary-level children have a lot of fun and use their own methods. Examining these methods, and observing how they try to solve the problem in their own way, their method also has mathematical reasoning and a logical approach to problem-solving.



SANDEEP DIWAKAR has been working as a mathematics resource person at Azim Premji Foundation, Bhopal, M.P. since 2012. He has experience teaching mathematics at higher secondary schools and worked in Rajya Shiksha Kendra (SCERT) Bhopal for 15 years as a lecturer. Sandeep has been associated with the development of SCE, textbooks, training modules and teaching learning material for teacher educators, teachers and children. His articles have been published in Shaikshik Palash, Prathmik Shikshak, Shaikshik Sandarbh, etc. Sandeep may be contacted at sandeep.diwakar@azimpremjifoundation.org

Measuring Javelin Throws with a Broken Tape

MOHAN R

A Solution:

Let A be the landing point, let α be the circular arc closest to A , and let O be the centre of α . To ensure a valid measurement, the tape must align with the line connecting A and O . Let C be the point of intersection of the line AO and the circular arc α . The physical education teacher, acting as the referee, needs to find the right spot, point C , on α . If the referee doesn't do this correctly, the measurement isn't fair, and the runner-up's point is valid. How can the referee make sure to find point C accurately?

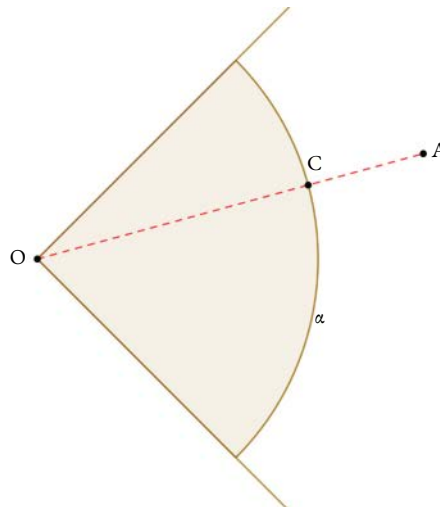


Figure 1

If we choose any other point B on α , by triangle inequality, we would have $OB + BA > OC + CA = OA$.

Keywords: Applications, inter-disciplinary, problem-solving.

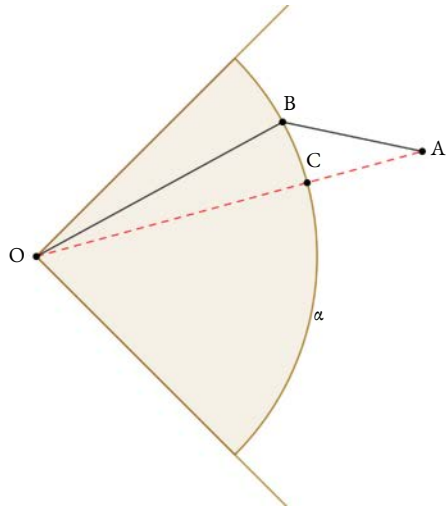


Figure 2

So, if B is a point on α different from C, we can draw another circular arc β using A as the centre and AB as the radius. Now β cuts α at the original point B and at a different point D (Will this happen if B and C coincide?).

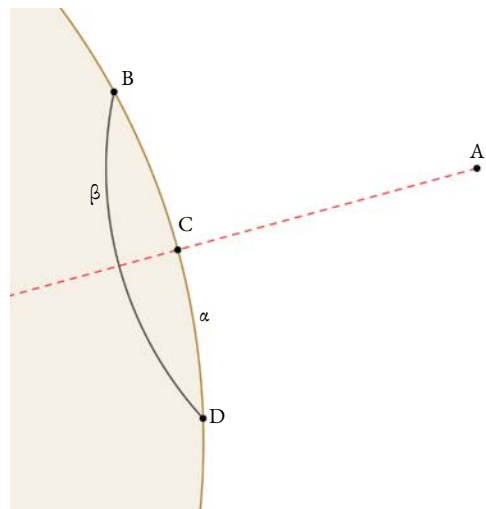


Figure 3

Let us look at the exaggerated figure given below. Let the line joining B and D intersect the line joining O and A at point M. Then the triangles BOA and DOA are congruent (by SSS). It follows that $\angle BOA = \angle DOA$, and hence the triangles BOM and DOM are congruent (by SAS). This implies $\angle OMB = \angle OMD$, making them right angles. Thus, lines BD and OA are perpendicular.

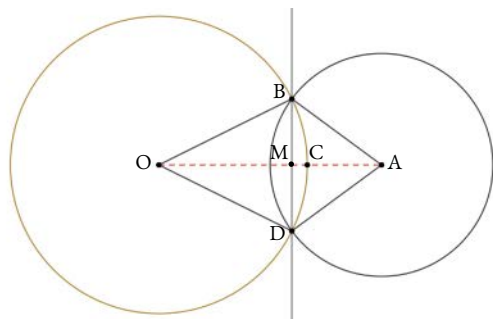


Figure 4

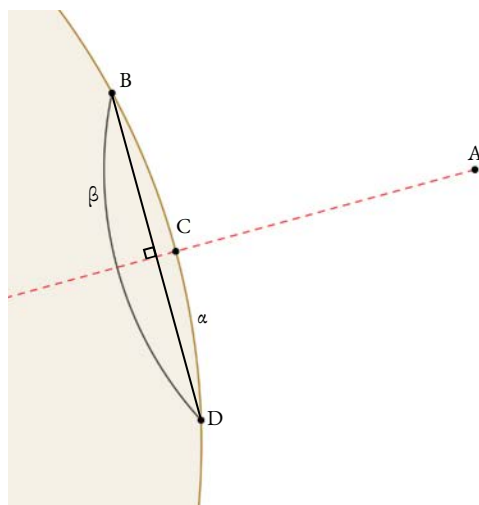


Figure 5

Hence, if the referee chooses any point B on the circular arc α , she can create a perpendicular bisector of BD (or angle bisector of $\angle BAD$) using basic geometric methods learned in middle school. The point where the bisector meets the arc α is the desired point C.

Editor's note: Geometry becomes more engaging when we connect it to real-world experiences. In sports like shot put, hammer throw, and discus throw, distance measurement follows the principles discussed here, particularly for reasoning, error analysis and justification.

Moreover, this discussion naturally leads to the concept of the *tangent of a circle at a point* on the circle. If the referee chooses point C instead of B, it's evident that the new circular arc β does not intersect α at any other point. Therefore, the line at C, perpendicular to AO, represents the tangent at C. This concept underpins the following theorem, typically introduced to students later:

Theorem: The tangent to a circle at any point is perpendicular to the radius of the circle that passes through the point of contact.

Acknowledgement

At Right Angles is grateful to Mr. Suyash Tiwari, mathematics teacher at Azim Premji School, Dhamtari. His write-up on measuring javelin throws seeded the idea for this article.



MOHAN R teaches mathematics at Azim Premji University. An algebraist by training, he is also interested in mathematics education and mathematics communication. He is the regional coordinator for the Mathematical Olympiad for Karnataka. He may be contacted at mohan.r@apu.edu.in

As dawn broke over the New Mexico desert on July 16, 1945, a historic silence pervaded the atmosphere as the first atomic bomb awaited its detonation, witnessed by a tense assembly of its creators.



The moment of truth arrived at 5:29 A.M. The first atomic bomb detonated, unleashing a blinding flash and a deafening roar, as the shockwave surged outward with unfathomable force.



Among the brilliant minds gathered there, Enrico Fermi, a key figure in the Manhattan Project, awaited the test. Moments before detonation, he prepared a simple experiment.



As soon as Fermi saw the flash of the bomb, he released some paper scraps above his head and observed as they fluttered down, landing a little over 2 metres away.



Within seconds of the explosion, Fermi had an estimate for the bomb's energy to ten thousand tons of TNT. Weeks later, meticulous analysis confirmed Fermi's estimate was remarkably close to the actual yield.

How could he 'guesstimate' such a large quantity in an instant?

The Art of Guesstimation - Part 1

MOHAN R

This article explores the Fermi method, a quick and effective way to estimate large numbers with a minimal number of guesses. In this first part, we'll demonstrate the method with a few examples. Stick around for a list of intriguing Fermi problems for you to solve at the end!

The Fermi Method

Enrico Fermi (1901–1954) was an Italian physicist who made significant contributions to the fields of nuclear physics and quantum mechanics. His groundbreaking research on nuclear reactions caused by slow neutrons earned him the Nobel Prize in Physics in 1938. Right after winning the Nobel Prize, he moved to the United States to flee Mussolini's fascist regime. Four years later, he had produced the first sustained nuclear reaction in Chicago. This discovery paved the way for the development of atomic bombs and nuclear fission reactors, revolutionizing our understanding of nuclear energy and its potential applications.

Fermi's intellectual prowess extended beyond his scientific pursuits. He possessed a remarkable ability to make accurate estimates and solve complex problems with limited information. His unconventional approach to problem-solving, often relying on intuition and common sense, became known as the "Fermi method" or "Fermi estimation."

Fermi often entertained his friends and students by creating and solving unusual problems, such as "How many piano tuners are there in Chicago?". A 'Fermi Problem' asks for a quick estimate of something that seems hard or impossible to measure accurately.

Keywords: Fermi Method, Estimation, Creative Problem-solving, Reasoning

Fermi's approach to these problems was to use common sense and rough estimates to get a general idea of the answer.

What are Fermi problems and how to solve them?

Let us consider, for instance, the problem of determining the number of piano tuners in the city of Chicago. How is this different from another estimation problem of determining how many seconds there are in a year? To solve the latter problem, we need to know the number of days in a year, the number of hours in a day, the number of minutes in an hour, and the number of seconds in a minute. All of them have a definite answer. Then we simply have to convert the unit of time into seconds per year.

$$\frac{365 \text{ days}}{1 \text{ year}} \times \frac{24 \text{ hours}}{1 \text{ day}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} \times \frac{60 \text{ seconds}}{1 \text{ minute}} \approx 3 \times 10^7 \frac{\text{seconds}}{\text{year}}$$

In some sense, the latter problem can be solved through logical deductions with the data given in the statement of the problem itself.

Fermi problems, on the other hand, differ markedly from the usual mathematical problem in that the answer to a Fermi problem cannot be verified by logical deductions alone and is always approximate. To know the exact number of piano tuners in Chicago, you may need to conduct a head count of all piano tuners in the city! All piano tuners may either not be listed in the telephone directory or cannot be found using a Google search.

Here's one way you could figure out how many piano tuners there are in Chicago:

1. Start by guessing how many people live in Chicago.
2. Guess how many homes there are in Chicago.
3. Estimate the fraction of homes that have a piano.
4. Guess how often each home gets their piano tuned.
5. Estimate how long it takes to tune one piano.
6. Guess how many hours a piano tuner works in a week.

When we're estimating, we break down the problem into smaller steps, and in each step, we have to make some guesses. Since we're aiming for a rough estimate, not an exact one, we just need to make sure our guess is within the right order of magnitude. We might overestimate or underestimate at each stage. For example, not only houses have pianos – public places and businesses might too. Some pianos might be tuned more or less often than we think. The law of probabilities says that if we make errors in estimates in different directions, they can balance each other out, and our final estimate will be closer to the right answer.

A similar solution for estimating the number of piano tuners in Chicago can be watched in the following TED-Ed video: [A clever way to estimate enormous numbers - Michael Mitchell](#)

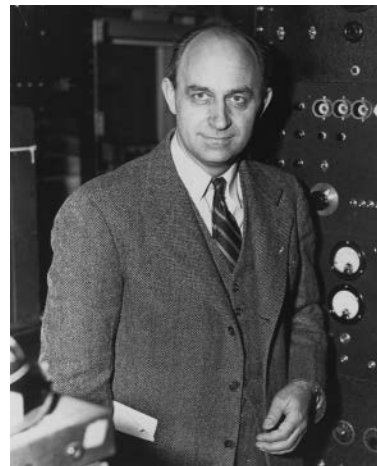


Figure 1. Portrait of Enrico Fermi.
Source: Enrico Fermi. (2024, January 26). In *Wikipedia*.
https://en.wikipedia.org/wiki/Enrico_Fermi

Here are some sample Fermi problems and strategies to solve them.

Problem 1: How many bald people are there in the world?



Figure 2. It is impossible to count the total number of bald people in the world

A solution: There are about 8 billion people in the world. About half of them are women, who don't typically get bald. So, that leaves us with about 4 billion men. Most people who get bald are over 30 years old. So, we can divide the 4 billion men into two groups: 2 billion who are 30 or younger and 2 billion who are older than 30. The older group is the one with the bald people. We can estimate that about 10% of the men in the older group are bald. That means that, by this reasoning, there are about 200 million bald people in the world.

Problem 2: How many bicycle repair shops are there in Bengaluru?

A solution: This problem is similar to the piano tuner problem. Bengaluru is a metro city in India and a typical metro city would have a population of 10 million people. If there are roughly 4 people in a household, there will be 2.5 million households in the city. If we assume that roughly one in two households own a bicycle, then there are approximately 1.25 million bicycles in the city. Assuming that a bicycle requires a repair once every year, there are 1.25 million repairs each year, which is roughly 100,000 repairs per month. Assuming that a repair shop can handle 50 repairs per month, to meet the demand, we need to have 2000 repair shops in the city. Note that we cannot be sure if the number of bicycle repair shops is 2000 or 6000, but we know that the order should be in thousands. So the number cannot be 200 or 20,000.

Here are some practical Fermi problems from different areas:

- In environmental policy: “If we stop using plastic grocery bags, how much less trash will we produce?”
- In educational policy: “If a state limits the maximum class size to 35 students, how much more money will it cost each year to run the schools?”
- In public health: “There’s a serious flu going around, and everyone in our country needs a vaccination from a health care worker. How fast can we get everyone vaccinated?”
- Personal finance: “A homemaker wants to start a morning tiffin shop to help with household expenses. Does she need to take a loan, and can she manage the business on her own?”

These examples illustrate the diverse range of applications for Fermi problems, demonstrating their usefulness in various fields where precise measurements are not always feasible. Here are more sample Fermi problems.

1. How many people in the world are talking on their mobile phones at this instant?
2. If everyone in your district donated one day’s wage to a good cause, how much money could be raised?
3. How many kilometres of roads/rivers are there in your state?
4. How much petrol does a typical motorcycle use during its lifetime?
5. How far does a butterfly fly each day?
6. What is the current population of mosquitoes in your city?
7. What is the average lifetime of a pencil?
8. How much does it cost to leave a tube light on for an entire week/month/year?
9. How many hours of TV will you watch in your lifetime?
10. How long will it take to count to one million? to ten million?
11. How much milk is produced in India each year?
12. Assuming that a suitable drawing surface could be placed along the entire route, how many pencils would it take to draw along the equator of the Earth?
13. If you posted an advertisement in a newspaper, how many people would be likely to see it?
14. How much food does your school consume in a month?
15. How many trees would need to be planted to lower the average global temperature by one degree? (assuming global warming is reversible)

There is no doubt that Fermi problems are fun to pose and solve, but can they be used in a classroom situation? Let us explore that question in Part 2 of the article.



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2D Base-10 Blocks

(also known as Flats-Longs-Units)

MATH SPACE

Many teaching-learning aids help one understand the base-10 structure and how we conceptualise and use whole numbers. Several of these also aid in introducing and exploring the four operations. We found that the best of these is 2D base-10 blocks

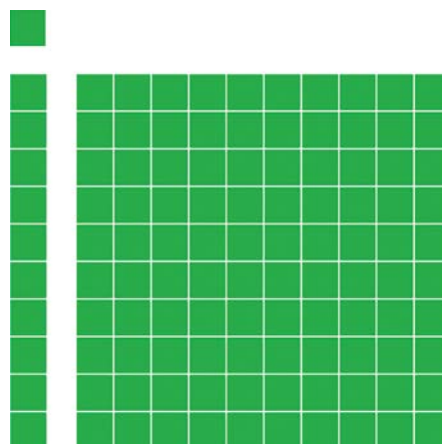


Figure 1

popularly known as Flats-Longs-Units (FLU). The unit is a small square or a 1. The long is ten times the unit and therefore a 10. Finally, the flat is a bigger square, a hundred times the unit and ten times the long, therefore a 100. Figure 1 illustrates these basic blocks. All three types of blocks should be of the same colour for the reason explained below.

Since this is a pre-grouped proportional material, the long cannot be unbundled into 10 units. So, it has to be exchanged for 10 units. Similarly, the flat has to be exchanged for 10 longs. Because of this pre-grouped nature, FLU cannot provide direct grouping and ungrouping experience, unlike a groupable

Keywords: Teaching Learning Materials (TLM), Place Value, Number Operations, Visualisation

material, for example, bundle sticks. Therefore, it is advisable to use FLU after exposure to some groupable material.

We recommend that for younger children, i.e., at the foundational level, preprimary and Class 1–2, a larger version made of corrugated cardboard (Figure 2) is provided. The sizes of the blocks can be:

- Unit: 2 cm × 2 cm
- Long: 20 cm × 2 cm
- Flat: 20 cm × 20 cm

If children at this stage are introduced to Thousand, then that can also be made by joining 10 flats using transparent cello-tape (Figure 3).

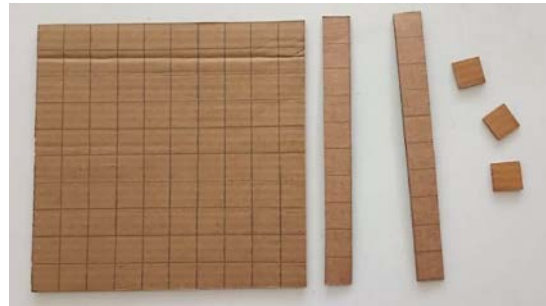


Figure 2



Figure 3

For older children, i.e., at the preparatory and middle stages, Class 3–5 and Class 6–8 respectively, smaller blocks made with square grid notebook paper pasted on thick chart paper or thick poster can be used (Figure 4).

FLU can be used for

- Comparing whole numbers.
- Addition-subtraction, especially, constructing the standard algorithms.
- Multiplication and division, both using the notion of the array.
- Squares and square roots – introduction and constructing the algorithm for finding the square root by long division.

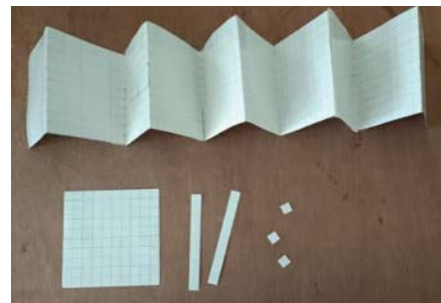


Figure 4

There should be an adequate amount of each block for each use:

- Addition-subtraction – at least 20 of each type
- Multiplication-division – at least 12–20 flats, 90 longs and 90 units
- Square and square root – similar to multiplication-division

When children start working with whole numbers (comparing them and using the four operations), their practice often reduces to symbolic manipulation without understanding the *whys* behind the rules

and algorithms. FLU fills that gap very well by matching numbers to the respective quantities they represent.

While comparing two whole numbers, FLU helps a learner understand that a single long is more than 9 units and similarly a flat is more than 9 longs (or 90 units). Therefore, any 2-digit number is greater than a 1-digit number and likewise, any 3-digit number is more than any 2-digit number. These observations can be generalized to “the whole number with more digits is larger”, e.g., $10002 > 98$. One can also conclude that if two 2-digit numbers have different leading digits, then they are represented with different quantities of longs. Naturally, the number with more longs is bigger, i.e., “if two numbers have the same number of digits, then the one with the bigger leading digit is larger”, e.g., $43 > 34$. The same conclusion can also help them to reason that $403 > 289$. And finally, if the leading digits are equal, then we should check the quantities of the next digits. For example, if two 3-digit numbers have the same leading digits, then we need to check the number of longs needed to represent each, i.e., the next digit on the right, e.g., $640 > 638$. If they are also the same, then check the next digits, e.g., $756 > 753$. Thus, arriving at the last rule, “if leading digits are also the same, check the next digit to the right, larger digit \Rightarrow bigger number” and “keep going till the digits differ”.

The column addition and ‘carry over’ or regrouping becomes automatic with a few simple ideas:

- Making each number (to be added) using FLU.
- Addition means to combine.
- Whenever, there are 10 of a kind, exchange with the next bigger block – i.e., if there are 10 or more units then exchange 10 units for a long, or if there are 10 or more longs then exchange 10 of those for a flat.

The teacher can just show how the algorithm is simply a way to record this. Figure 5 illustrates this for $487 + 376$.

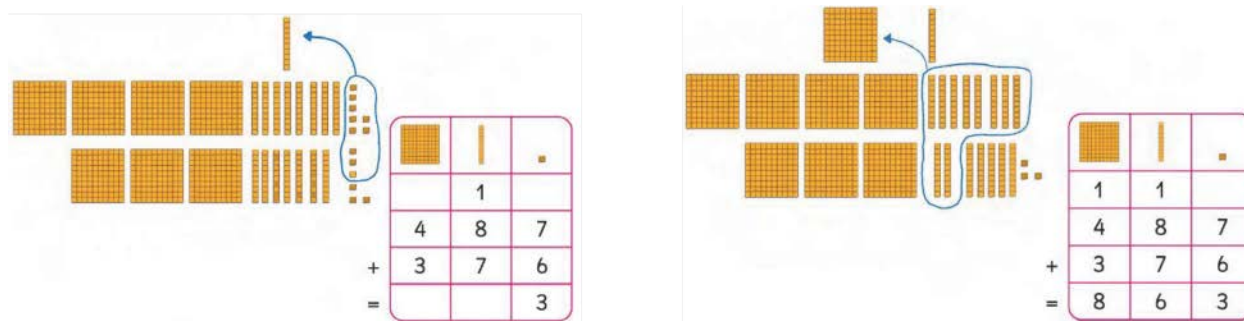


Figure 5

A similar process for subtraction would be:

- Making the first number using FLU.
- Visualising the second number, especially the FLU needed to make it.
- Subtraction means ‘to take away’.
- Exchange longs and flats, if needed, so that there is enough of each kind to take away.

Again, the teacher can facilitate to show how the standard algorithm just writes this down. Figure 6 illustrates this for $300 - 137$, a difficult subtraction demanding two rounds of exchange at the very beginning!

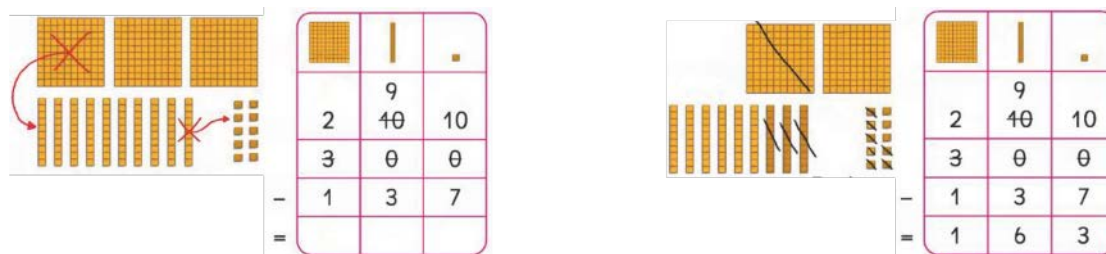


Figure 6

We have already illustrated division in the article on the Division of Whole Numbers. Figure 7 shows a multiplication example (4×15) and a second example for division ($779 \div 31$).

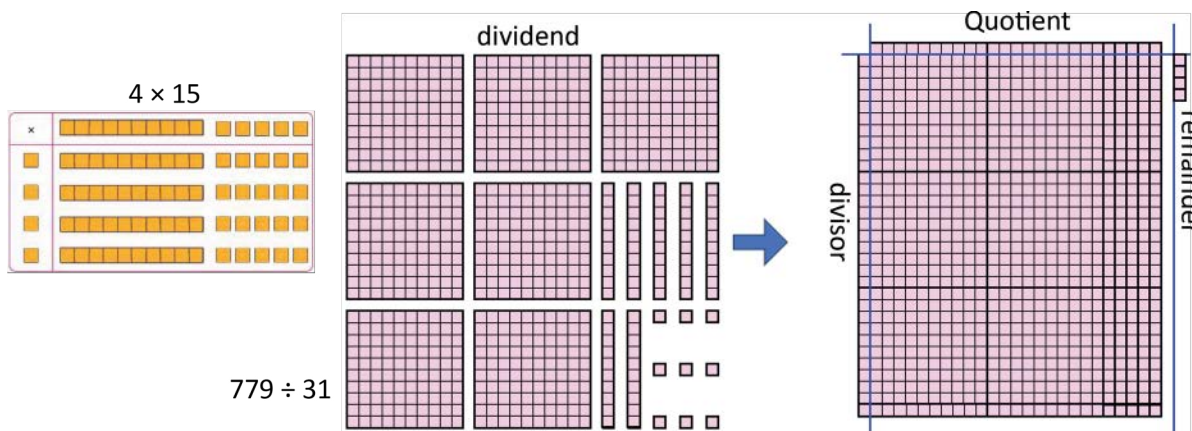


Figure 7

One usually encounters such arrays while making FLU – both larger and smaller versions. Figure 8 illustrates the material needed to illustrate 14×12 , i.e., 1 flat, $4 + 2 = 6$ longs and $2 \times 4 = 8$ units that can be made from one sheet of square grid.

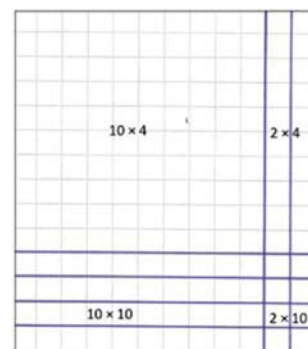


Figure 8

Squares can be considered special cases of multiplication. Squares of 2-digit numbers, in particular, resemble the pattern of $(a + b)^2$. This very identity is used in the division algorithm to find square roots. And this can be initiated using FLU. However, since this is done much later, with much older learners, the need for physical manipulatives would be less. But one can use the idea to draw pictures and rough diagrams.

Many virtual manipulative sites, including Mathigon Polypad, provide virtual FLU, which is as good as physical manipulatives. However, these come in different colours. But fortunately, one can change the colours as well. Figure 1 was generated using polypad.

FLU can be extended to decimals easily since it is base-10. However, there are some crucial changes.

- The flat becomes 1, whole, and therefore, should not have any lines within.
- The longs are $1/10^{\text{th}}$ of the flat, i.e., 0.1 and should not have any lines either.
- The units are $1/10^{\text{th}}$ of the long, so, $1/100^{\text{th}}$ of the flat, i.e., 0.01.

We recommend no lines so that learners see the big square in decimal FLU as one or whole. If it includes lines, then learners would be prone to count, and it will become 100 instead of 1. Since learners would be older when they start working with decimals, physical manipulatives would be needed less, and more focus should be on drawing on square grid notebook, and later centimetre graph paper with:

- 10 cm \times 10 cm square as 1
- 10 cm \times 1 cm rectangle as 0.1
- 1 cm \times 1 cm square as well as 10 cm \times 1 mm rectangle as 0.01 – the latter is very useful in decimal multiplications such as 0.34×0.27
- 1 cm \times 1 mm rectangle as 0.001
- 1 mm \times 1 mm square as 0.0001

Addition and subtraction are almost identical. Multiplication also resembles the corresponding whole number counterpart quite closely. Figure 9 illustrates 0.14×0.12 on graph paper, which is very similar to 14×12 .

FLU also generalizes to algebra tiles as a base, i.e., 10 is replaced by the variable x . We will discuss algebra tiles in a later issue. But it is worth mentioning here that the processes, especially w.r.t. multiplication and division are eerily similar!

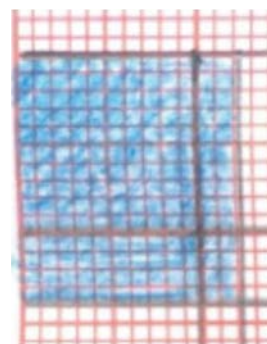


Figure 9

Acknowledgement

Anupama S M of Azim Premji University extended the usual FLU to include a thousand.

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MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

A Call for Articles

At Right Angles is a quality resource dedicated to mathematics education in India's public education system. It is specifically designed for teachers and teacher educators at the foundational, preparatory, and middle school levels.

We invite articles from mathematics teachers, educators, practitioners, parents, and students. If you are looking for a platform to contribute articles that support and enhance the learning experience of mathematics particularly for students approximately in the age group 6-14 years, we welcome your submissions.

SUGGESTED TOPICS AND THEMES

Submitted articles should focus on curricular content applicable to Classes 1-8 and could:

- Explain and illustrate themes and topics outlined in the National Curriculum Framework for School Education 2023 (NCF-SE 2023).
- Specifically address challenges discussed in the NCF-SE 2023.
- Be substantiated accounts of the history of mathematics or the history of mathematical thinking.
- Include innovative worksheets or methods to engage students in drill and practice.
- Describe real-life applications of mathematics relevant to the child's context.
- Describe interdisciplinary activities or projects.
- Review puzzles or games with a practical connection to the syllabus.
- Offer guidance on selecting relevant content, including online resources.

- Develop pedagogical strategies for foundational numeracy as well as computational thinking.
- Assist teachers in implementing differentiated teaching practices.
- Review of Teaching Learning Material (TLM) or describe how to use local context, and local TLM in the math class.
- Provide material to help students bridge gaps in conceptual understanding.
- Address issues in assessment.
- Suggest ways to identify and address misconceptions in mathematics learning.
- Offer a list of problems along with discussions on their solutions and problem-solving strategies that are not commonly found in textbooks.

In addition to full-length articles, we also welcome shorter pieces that can include a variety of engaging content. These could be reviews of books, mathematics software, or YouTube clips that explore mathematical themes. Other contributions can be 'proofs without words', mathematical paradoxes, 'false proofs', or creative expressions such as poetry, cartoons, or photographs with a mathematical theme. We also welcome anecdotes about a mathematician or interesting examples of 'math in craft, movies, etc.'

Articles may be sent to

AtRightAngles.editor@apu.edu.in

Please refer to specific editorial policies and guidelines on the following page.

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The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be indicated in the article.

At Right Angles brings out translations of the magazine in other Indian languages. Hence, Azim Premji University holds the right

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1. **Engaging Introduction:** Write in a readable and inviting style, aiming to capture the reader's attention from the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, a figure with an interesting question, or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. **Catchy Title:** Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. **Style:** Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. **Balance:** Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps that depend on hidden calculations.
5. **Accessible language:** Avoid specialized jargon and notation that will be familiar only to specialists. If technical terms are needed, please define them.
6. **Use visuals:** Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. **Concise References:** Provide a compact list of references, with short recommendations.
8. **Exercises and Questions:** Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. **Citation format:** Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
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11. **Labelling visual elements:** Label and number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note: the minimum resolution for photos or scanned images should be 300 dpi).
12. **Precise references to visuals:** Refer to diagrams, photos, figures and tables by their numbers and avoid using references of these kinds: 'here', 'there', 'above', 'below', 'to the left', 'to the right'.
13. **Author Bio:** Include a high-resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. **British Spelling:** Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. **Format for submission:** Submit articles in MS Word format or in LaTeX.

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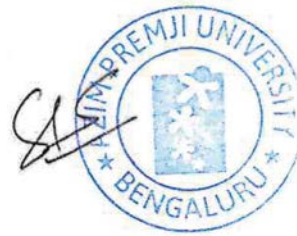
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MASTERY OF MULTIPLICATION

PADMAPRIYA SHIRALI

MASTERY OF MULTIPLICATION

When do we know that a student has mastered a particular concept well? An ability to repeat a learnt process or a practised algorithm is not a necessary indication of mastery. One looks for the ability to apply the concept in situations or contexts, where the concept is embedded indirectly.

Here are a few contexts and problems (not sequenced in any order) which can be presented to students to assess their understanding of the application of the multiplication process. Some involve pure reasoning, while others involve pattern recognition, understanding relationships, multiplication shortcuts, counting, etc.

Problems that involve reasoning require a deep understanding of the concept, in order to handle abstract numbers and operations on them in a flexible manner. These problems are context dependent and cannot be turned into formulae.

Problems that require investigation into the patterns and relationships generated by the multiplication process aid in developing multiplication shortcuts. They help to develop a range of mental strategies. Multiplication shortcuts are used in mental computation which is a part of everyday life.

A problem that involves counting can be presented either through actual models or pictures. There are no standard ways of looking at these models, and they bring forth each student's facility in visualisation.

It is important to keep in mind that the purpose of these activities is not to solve problems but to develop a curiosity and a desire to discover different strategies.

Students may use materials, and drawings to aid in their understanding of the problem situation.

We would recommend that all these problems be solved initially by students working independently. It can then be followed by discussions among the students about the different approaches that they have used to solve them. This provides exposure to the varied ways a problem can be perceived and approached. Let students attempt the strategies tried out by others.

An important learning for the teacher in this process is to become aware of the level of comfort that students have with the types of computations. It also gives scope for teachers to become aware of the thought processes and reasoning employed by the students.

Prerequisite: All these activities presume that students have acquired a basic knowledge of multiples, factors and prime factorisation. Hence, they can be used at the level of class 5 or class 6.

Keywords: Multiplication, Pattern Recognition, Strategizing, Conceptual Understanding

PROBLEM 1

Objective: Logical reasoning

Material: Flashcards

If $6 \times 10 = 60$, what is 12×5 ?

Students may know the multiplication facts of 12×5 and see that the answer is the same.

Do they see the relationship between the two sets, i.e., 6×10 and 12×5 ?

Can they explain why the product turns out to be the same?

What is the effect of halving one factor and doubling the other factor?

They could be asked to build more such pairs for verification.

Can the students build another pair of factors that relates to the pair 6×10 ?

30×2 is another such pair. How does it relate to the pair 6×10 ?

Are the students able to see that 2 is *one-fifth* of 10, and 30 is *five* times 6?

It is good if students notice that the situations are structurally similar. In the first example, the factors got doubled and halved. In the second example, one factor became 5 times while the other became one-fifth.

If $100 \times 9 = 900$, what is 25×36 ?

How are these two pairs related?

Can the students build other factor pairs that relate to the pair 25×36 ?

What strategies do the students use to solve the problems?

Can the students create more examples to demonstrate this principle?

In this problem and others that follow, one can see the linkage with factors, multiples and prime factorisation.

$$6 \times 10 = 60$$

$$12 \times 5 = ?$$

$$100 \times 9 = 900$$

$$25 \times 36 = ?$$

PROBLEM 2

Objective: Investigation through arrays to discover the doubling and halving strategy of multiplication.

Material: Peg-board or dot-sheet

Here is a visual for how 4 rows and 3 columns are rearranged to form 2 rows and 6 columns.

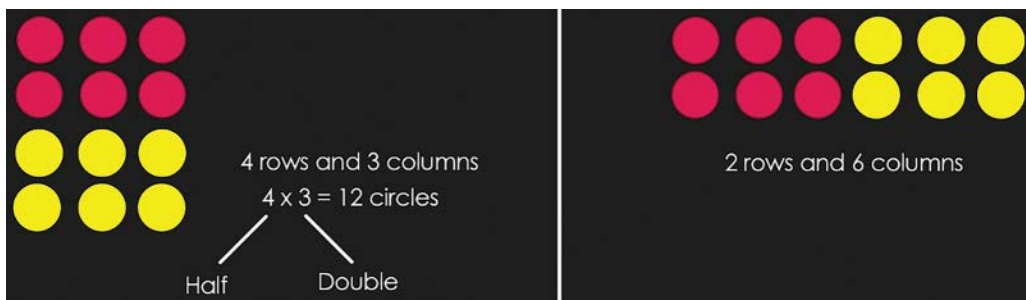


Figure 1

Students can now be asked to build an array to demonstrate what 8×6 looks like.

Let them rearrange the pegs as other possible rectangular arrays. How do the other pairs relate to the original pair? i.e., 8×6 .

2×24 (2 is one-fourth of 8 and 24 is 4 times 6)

3×16 (3 is half of 6 and 16 is twice 8)

4×12 (4 is half of 8 and 12 is twice 6)

What do they observe and conclude?

In the case of both the 3×16 and 4×12 arrangements, the array has been rearranged by halving one factor and doubling the other.

The halving and doubling strategy involves halving one of the factors and doubling the other.

For example, for 15×24 , we can double 15 to make 30 and halve 24 to make 12.

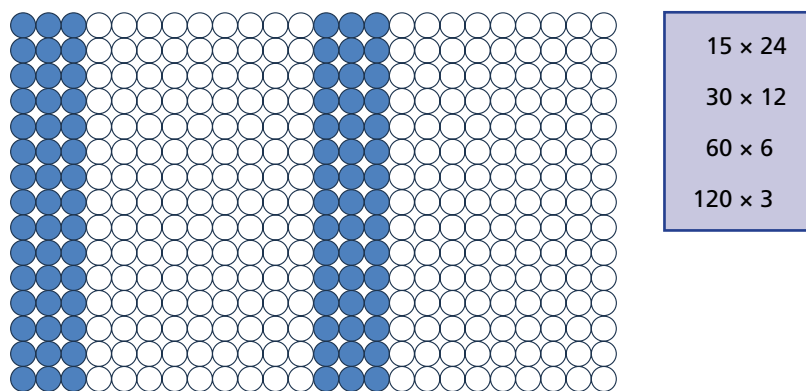


Figure 2

This process can continue till the multiplication becomes easier. 30 can be doubled to make 60 and 12 can be halved to 6.

60×6 is easy, that is, 360.

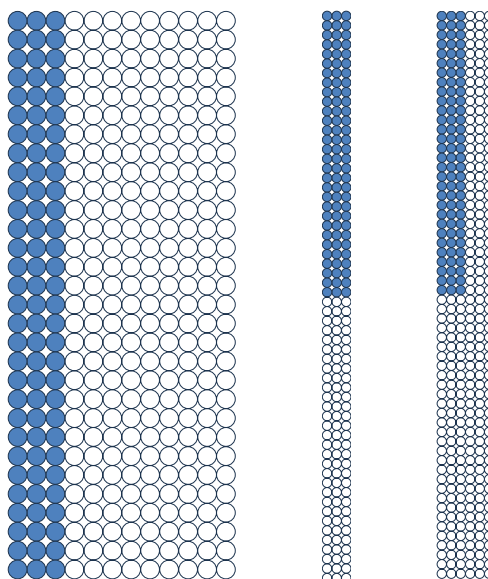


Figure 3

Discuss how this process is helpful in doing multiplication. Let students try the approach for some problems to notice its efficacy.

For what problems does the doubling and halving approach work well?

Let them experiment with more such arrays, e.g., 6 rows, 7 columns, etc. (Where the number of rows is even but the columns are odd.)

Will it work well for 11×13 ? Why would it not work well for this problem?

Will it work if one of the numbers is even?

Let students create some problems where such an approach makes the problem simpler to solve.

Here is one more problem where factorisation by 5 simplifies the problem.

e.g., $375 \times 28 = 75 \times 140 = 15 \times 700$, etc.

PROBLEM 3

Objective: Apply understanding of the concept, the laws of associativity, distributivity, etc.

PROBLEM 3.1

Here is a visual demonstration for 8×9 .

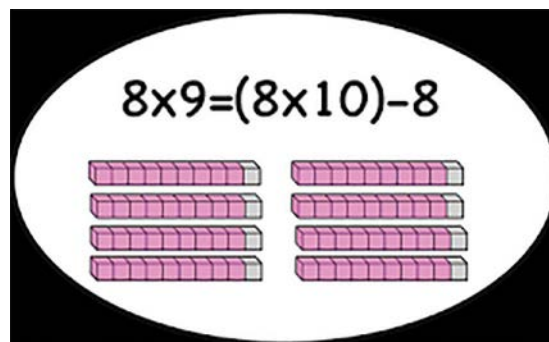


Figure 4

How would the student modify the approach to calculate 18×9 or 98×9 ?

PROBLEM 3.2

Are the students able to compute quickly using their understanding of the distributive law? 53 is 3 more than 50. They need to add 9×3 i.e., 27 to the given product.

$$9 \times 50 = 450$$

$$9 \times 53 = ?$$

$$9 \times 53 = 9 \times 50 + 9 \times 3$$

The student's choice of strategy depends on the facility they have with number facts. Strategies are bound to vary.

Problem 3.3

$$7 \times 8 = (5 + 2) \times 8,$$

$$6 \times 7 = (5 + 1) \times 7,$$

$$9 \times 7 = (10 - 1) \times 7,$$

$$8 \times 6 = (10 - 2) \times 6$$

Problem 3.4

Usage of associative property

$$8 \times 9 \times 10 \times 11 \times 12$$

What strategies will the students use to solve this problem?

Will they regroup the numbers as $8 \times 12 \times 9 \times 11 \times 10$?

$$8 \times 12 = 96 \text{ and } 9 \times 11 = 99$$

The problem has changed to $96 \times 99 \times 10$

Multiplying by 99 can be seen as multiplying by $(100 - 1)$.

$$(96 \times 100 - 96 \times 1) \times 10$$

$$(9600 - 96) \times 10$$

$$9504 \times 10 = 95040$$

Problem 3.5

Associative property is being used here.

$$11 \times 12 = 132$$

$$66 \times 12 = ?$$

Problem 3.6

$$600 \times 15 = 9000$$

$$600 \times 45 = ?$$

Problem 3.7

How would the students think about approaching these two problems? Discuss the strategies used.

$$\text{What is } 128 \times 8?$$

$$\text{What is } 26 \times 17?$$

The strategies for these 2 problems may be different.

A problem such as 128×8 can be attempted in different ways.

$$128 \times 8 = 256 \times 4 = 512 \times 2 = 1024 \times 1$$

or

$$128 \times 8 = 128 \times (10 - 2) = 1280 - 256 = 1024$$

Let the students make up similar problems and pose their problems to each other. Encourage them to explain their answers to each other.

PROBLEM 4

Objective: Reasoning out a problem

Pose problems that require reasoning to solve.

Problem 4.1

Two adjacent boxes will represent a double digit number.

Have the students used their understanding of place value?

Do they find more than one solution to this problem?

Position the digits 3, 4 and 5 to make the product as large as possible:

$$\square \square \times \square =$$

Problem 4.2

In what way are these problems alike and in what way are they different?

Fun problem: What is the product of the ten one-digit numbers?

If $4 \times 6 = 24$ what is $4 \times 600 = ?$

$$400 \times 6 = ?$$

$$40 \times 60 = ?$$

$$4000 \times 0.6 = ?$$

Problem 4.3

Here is another product problem that requires the use of logic in the substitution of letters a, b, c, \dots with numbers. Each letter stands for a single-digit number.

\times	a	b	c
d	12	\square	36
e	18	\square	54
f	\square	56	\square

Figure 5

Does this problem have a single solution or more than one solution?

Problem 4.4

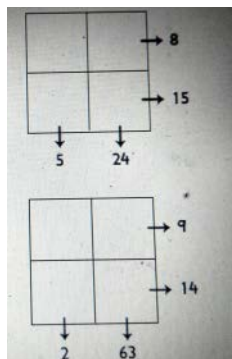


Figure 6

A nice problem where the products are given, and the grid has to be filled with the correct numbers to make the correct products (given on the right and at the bottom).

Problem 4.5

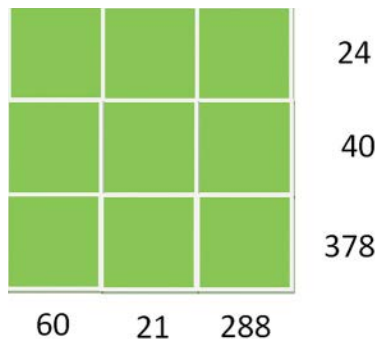


Figure 7

Here is another nice problem from NRICH. (<https://nrich.maths.org/11750>)

I like the fact that one can deduce the answer with logic and there is hardly any trial and error. It is a good way of reinforcing factors and multiple properties.

Use all the numbers from 1 to 9 in the grid to obtain the given products.

PROBLEM 5

Objective: Discover new relationships.

Let students explore patterns in multiplicative relationships in a series of numbers.

6, 7, 8, 9, 10, 11, 12

What is the relationship of 8×10 to 9×9 ? (notice that 8 and 10 are 1 step away from 9)

$8 \times 10 = 80$ which is 1 less than 81.

What is the relationship of 7×11 to 9×9 ? (notice that 7 and 11 are 2 steps away from 9)

$7 \times 11 = 77$ which is 4 less than 81.

What is the relationship of 6×12 to 9×9 ? (notice that 6 and 12 are 3 steps away from 9)

$6 \times 12 = 72$ which is 9 less than 81.

The discovery of this relationship can be later connected with $a^2 - b^2 = (a + b)(a - b)$

Can the students guess how 5×13 relates to 81?

What other observations can they make?

We see that pairs of numbers which are closer together have a greater product.

Now, can students use the fact that $45 \times 45 = 2025$ to figure out 41×49 ?

Can they explain how this can be done and find the product?

$$45 \times 45 = 2025$$

$$41 \times 49 = ?$$

One can build extensions to this discovery by posing further problems.

What is 197×197 ?

Can the students use rounding in this situation? 200 is 3 more than 197. The students can turn the problem into 200×194 (shifting by 3 on both sides) which is 38,800. Now they can add

$3 \times 3 = 9$ to the final number to get 38,809.

Let students work with other results and pose problems to one another.

PROBLEM 6

Objective: Multiplication in contexts

Problem 6.1

How many yellow circles?

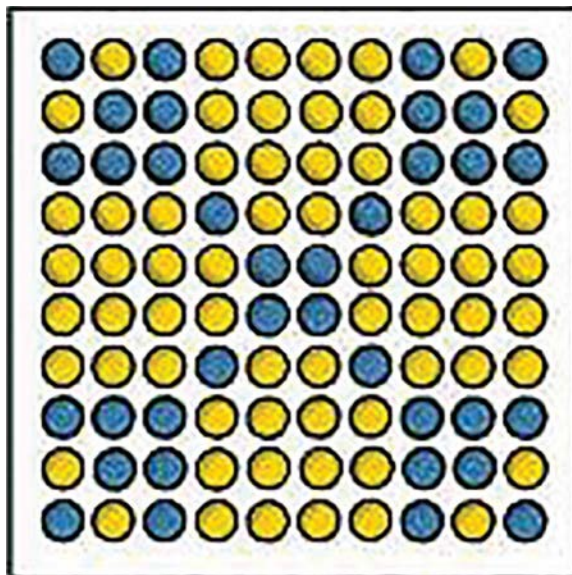


Figure 8

Problem 6.2

How many green squares?

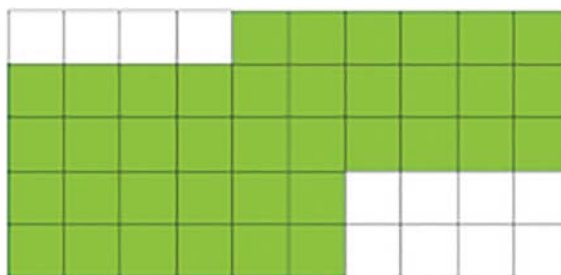


Figure 9

Problem 6.3

How many purple rectangles?

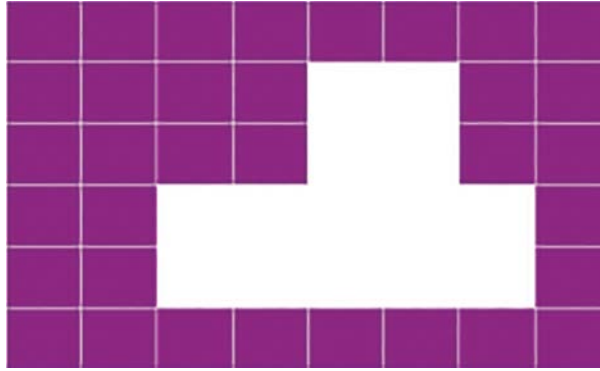


Figure 10

Note the relationship of these problems to area problems.

Problem 6.4

How many seats are on this train?

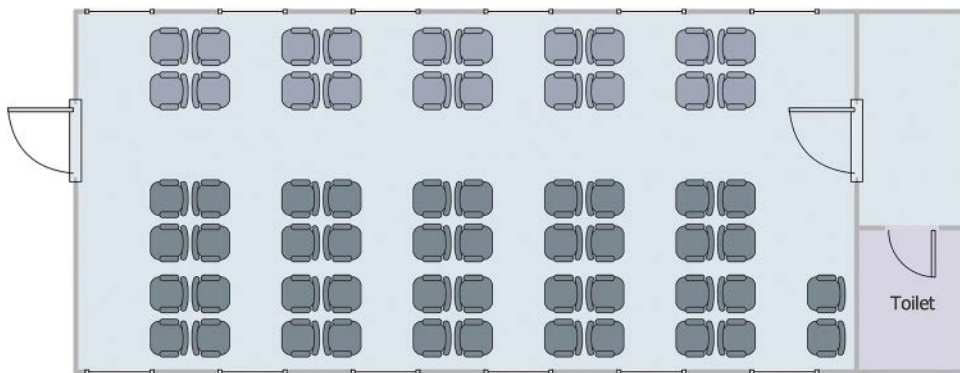


Figure 11

Problem 6.5

How many seats are on this flight?



Figure 12

PROBLEM 7: PEGBOARD ARRANGEMENT

Objective: Using the concept of multiplication in patterns

Here are some pegboard arrangements which children can create and use for counting.

How many pegs have been used in each collection?

Teachers should encourage students to share their different approaches.

Problems can be linked with area.

Here are two patterns where colour may also be used as part of the strategy.

Problem 7.1

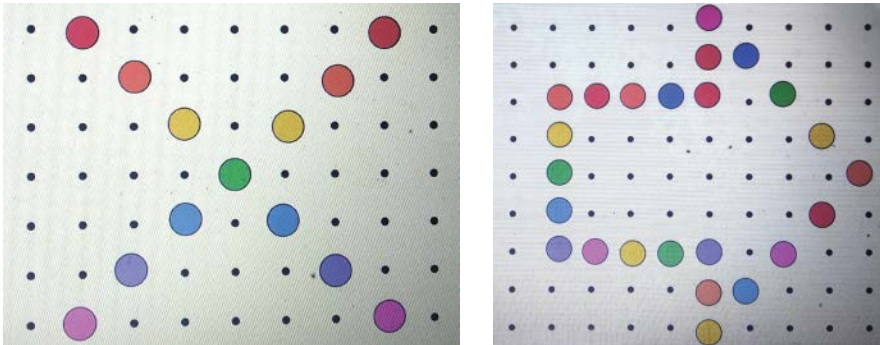


Figure 13

Problem 7.2

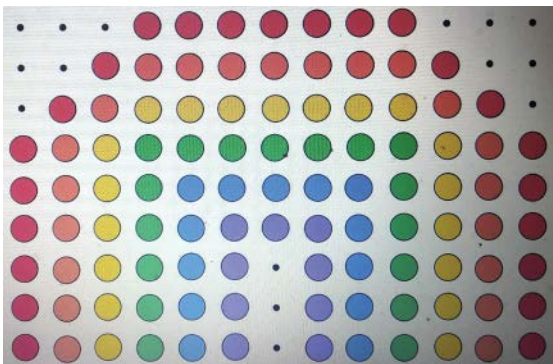


Figure 14

Problem 7.3

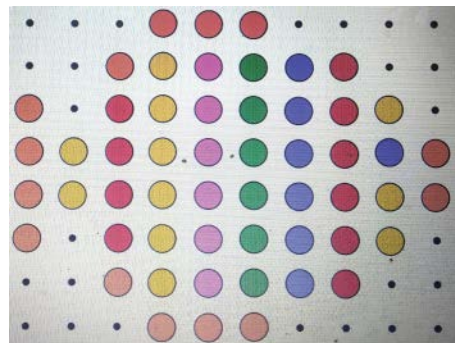


Figure 15

PROBLEM 8: RANGOLI DOTS AND MULTIPLICATIONS

Objective: Using the concept of multiplication in counting

Here is a set of dots made to create a rangoli design.

How many dots has the artist used?

What strategies do the students use for counting?

Let each student figure out a solution and share their strategies.

Would one strategy be to count for one triangle and the central hexagonal shape separately? How would the counting happen for each triangle?

Will the pattern 1, 2, 3, ... 7 be considered for summation?

What multiplications are used in the summation of $1 + 2 + 3 + 4 + 5 + 6 + 7$?

$(1 + 7) + (2 + 6) + (3 + 5) + 4$. There are three 8's and one four. $24 + 4 = 28$.

There are 6 triangles with 28 dots in each. That is 168 dots in the triangles.

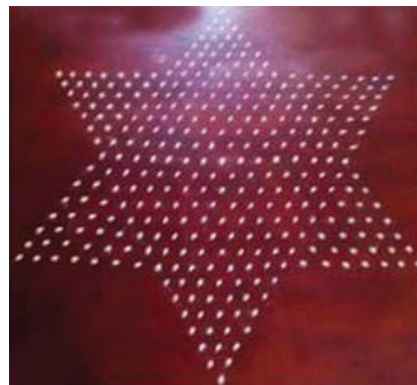


Figure 16

Would the hexagon be counted starting from the diagonal 15, 14, 13, ... 8? to get the number of dots for half of the figure.

$15 + 14 + 13 + 12 + 11 + 10 + 9 + 8 = (15 + 8) + (14 + 9) + (13 + 10) + (12 + 11)$ which is four 23's. That is, 92 dots in half the figure.

The full hexagon has 184 dots.

Altogether this design holds $184 + 168 = 352$ dots!

Another strategy may be to use the symmetry of the figure to work out half the dots. The dots are receding from 22, 21, 20, ... to 15 with a triangle shape on top.

Are there other ways of counting?

If you had to copy the design, how would you start?

Discuss your strategies and have fun making designs with it!

Here are two more designs for counting.

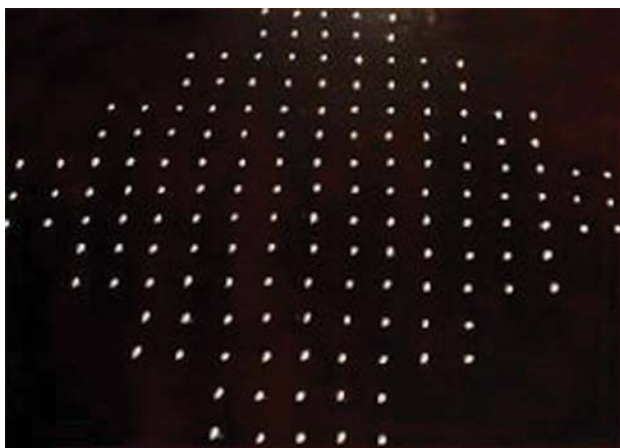


Figure 17

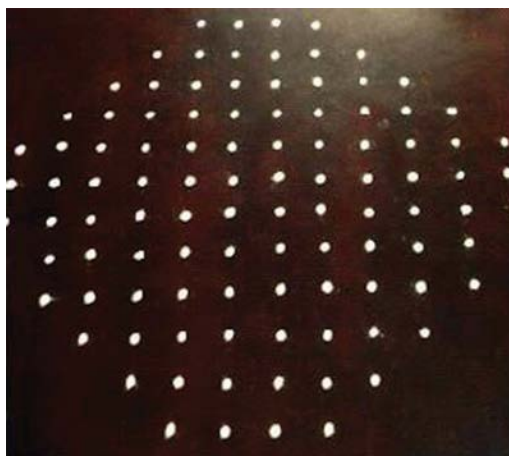


Figure 18

PROBLEM 9: CONSTRUCTIONS WITH CUBES AND MULTIPLICATIONS

Objective: Using the concept of multiplication in counting

Models like these can be made with Jodo cubes or virtually in Mathigon Polypad or <https://toytheater.com/cube/>

Let students begin with simple cube designs to explain their strategies of counting cubes.

How many cubes?

Most will perhaps count them as $6 + 1$ i.e., $(2 \times 3 + 1)$.

Problem 9.1

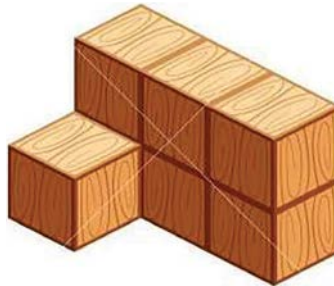


Figure 19

Teachers can connect this problem with volume.

Problem 9.2

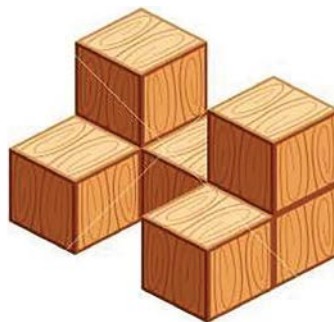


Figure 20

Problem 9.3

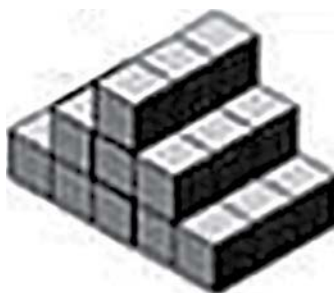


Figure 21

Will this be counted in horizontal layers or vertical slices?

Problem 9.4

How many cubes?

How will they approach this problem?

Is it easier to count the missing ones and subtract from the whole?



Figure 22

Problem 9.5

How many cubes in this E shaped construction?



Figure 23

Did the students resort to counting one by one? Or did they use three rows as 3 fours with 2 extra projections and 2 rows with one cube in each?

Problem 9.6

How many cubes?

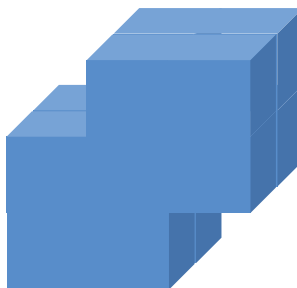


Figure 24

This is an interesting piece for generating discussion.

Some may choose to count them as 2 cubes of size $(2 \times 2 \times 2)$ with an overlap to be deducted. Or will they count in layers?

Problem 9.7

How many cubes?

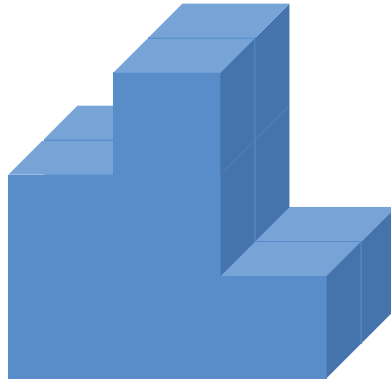


Figure 25

Again, discuss the strategies used.

Here are a few more such examples.

Problem 9.8

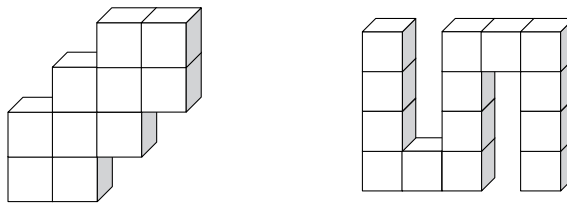


Figure 26

Problem 9.9

What approaches would the students take in counting the cubes in these constructions?

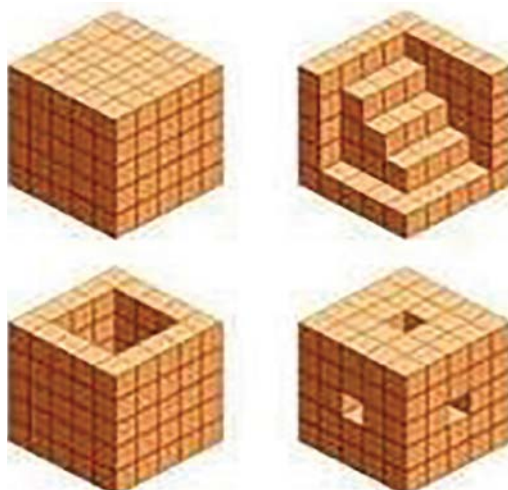


Figure 27

PROBLEM 10

Objective: Prediction of product in multiplication with decimal/fractional numbers

There is often a misconception that many students carry that multiplication always results in a bigger product. Pose problems which require them to estimate and not calculate to check their understanding.

23×0.2

23×2.4

543×0.62

65×0.7

864×1.2

98×0.65

Are the students able to predict where the answers lie?

Will the answer be less than the number or more? Can they give reasons for their thinking?

PROBLEM 11

Objective: Understanding the size of the products

A multiplication grid made to scale may serve as an additional help to students in visualising the numbers and multiplicative relationships.

What patterns do you see in the multiplication table?

Which shapes are squares and which are rectangles?

Does the grid aid to see why 7×9 is one less than 8×8 ? Or why 4×8 is less than 6×6 ?

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100
11	22	33	44	55	66	77	88	99	110
12	24	36	48	60	72	84	96	108	120
13	26	39	52	65	78	91	104	117	130
14	28	42	56	70	84	98	112	126	140
15	30	45	60	75	90	105	120	135	150
16	32	48	64	80	96	112	128	144	160
17	34	51	68	85	102	119	136	153	170
18	36	54	72	90	108	126	144	162	180
19	38	57	76	95	114	133	150	168	190
20	40	60	80	100	120	140	160	180	200

Figure 28

Acknowledgement:

<https://www.stem.org.uk/resources/elibrary/resource/32124/multiplication>

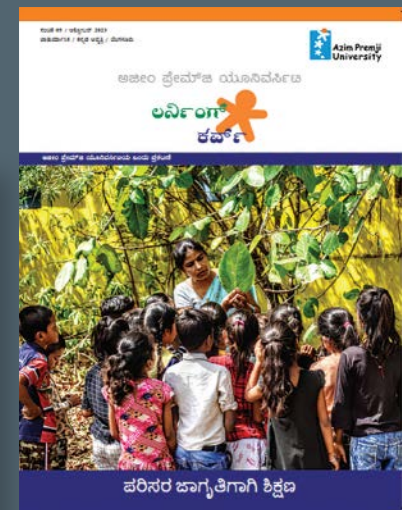
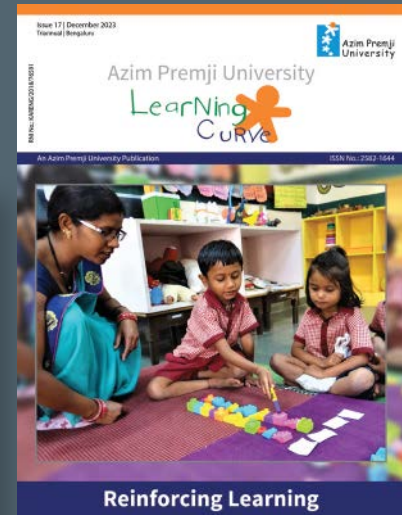
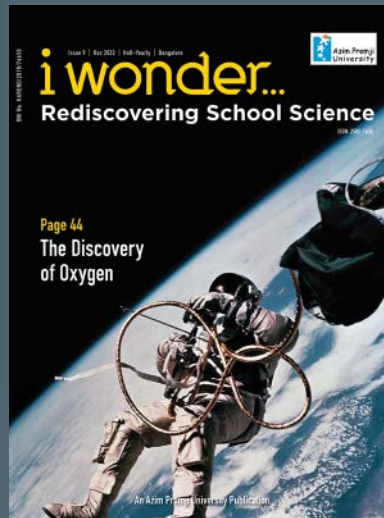
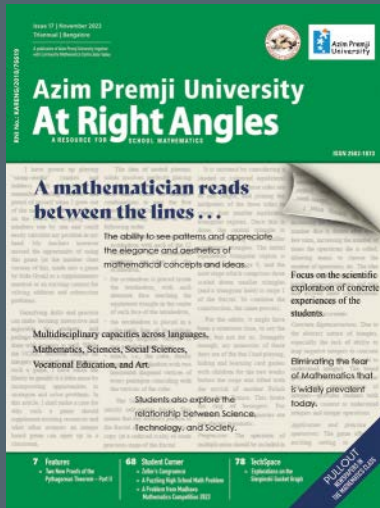
<https://stevevyborney.com>



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