Student Corner

A Problem from Madhava Mathematics Competition 2023

PARINITHA M

In this article, we discuss a solution to a number theory problem from Madhava Mathematics competition, 2023. (Madhava Mathematics Competition is a competition in mathematics for undergraduate students organized jointly by the Department of Mathematics, S.P. College, Pune, and Homi Bhabha Centre for Science Education, TIFR, Mumbai.)

Problem. Find all triplets (x, y, z) of non-negative integers satisfying the condition

$$x^2 + y^2 + z^2 = 16(x + y + z). \tag{1}$$

Solution. Let (x, y, z) be a triplet of non-negative integers satisfying condition (1). The condition implies that $x^2 + y^2 + z^2$ is even. Hence either two of x,y, z are odd and one is even, or all of them are even.

Since any odd square is of the form 1 (mod 4), if two of x, y, z are odd and one is even, then $x^2 + y^2 + z^2$ is of the form 2 (mod 4). But $16(x + y + z)$ is clearly a multiple of 4, so this possibility is ruled out. Hence all of x, y, z are even.

Let $(x, y, z) = 2(a, b, c)$ where a, b, c are non-negative integers. Substituting these into (1) we get $4a^2 + 4b^2 + 4c^2 = 32(a + b + c)$, hence

$$a^2 + b^2 + c^2 = 8(a + b + c). \tag{2}$$

Keywords: Parity, properties of square numbers, completing the square

The parity argument may be repeated and we see that all of a, b, c are even.

Let $(a, b, c) = 2(l, m, n)$ where l, m, n are non-negative integers. Substituting these into (2) we get

$$l^2 + m^2 + n^2 = 4(l + m + n). \quad (3)$$

The parity argument may be used yet again and we see that all of l, m, n are even.

Let $(l, m, n) = 2(r, s, t)$ where r, s, t are non-negative integers. Substituting these into (3) we get

$$r^2 + s^2 + t^2 = 2(r + s + t). \quad (4)$$

The parity argument may be used yet again and we see that all of r, s, t are even.

Let $(r, s, t) = 2(u, v, w)$ where u,v,w are non-negative integers. Substituting these into (4) we get

$$u^2 + v^2 + w^2 = 2(u + v + w). \quad (5)$$

This may be written, by "completing the square," as

$$(u^2 - 2u + 1)+(v^2 - 2v + 1)+(w^2 - 2w + 1)=3, \quad (6)$$

i.e., as $(u - 1)^2 + (v - 1)^2 + (w - 1)^2 = 3$.

Now, the only ways that 3 can be written as a sum of three squares are:

$$(\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 = 3 \quad (7)$$

It follows that

$$u-1 = \pm 1, v-1 = \pm 1, w-1 = \pm 1, \quad (8)$$

so $u = 0$ or 2; $v = 0$ or 2; $w = 0$ or 2. Any combination of these values leads to a solution of $u^2 + v^2 + w^2 = 2(u + v + w)$. Since $(x, y, z) = 8(u, v, w)$, we deduce that the solutions of the original equation are $x = 0$ or 16; $y = 0$ or 16; $z = 0$ or 16. Any combination of these values leads to a solution of (1).

Hence the solutions of the given equation are all the permutations of (0, 0, 0), (0, 0, 16), (0, 16, 16), and (16, 16, 16).

PARINITHA M is a II year BSMS student at IISER, Tirupathi. She is deeply interested in Mathematics, especially Analysis, Number Theory, and Group Theory. She took part in the Simon Marais Math Competition 2022 and the Madhava Mathematics Competition 2023. She is a recipient of the DST Inspire scholarship. She is an active member of The Math Club at IISER and hopes to pursue educational activities related to mathematics. Her interests include classical instrumental music. She may be contacted at parinitha.m.17@gmail.com