Student Corner

A Puzzling High School Math Problem

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Introduction

High-school students may have seen problems related to infinite continued fractions such as:

Find the value of m, where,

$$m = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}.$$

Motivated by the repeating structure we substitute the value of m, in the right-hand side of the equation. So, we get

$$m = 1 + \frac{1}{1+m}, \quad \therefore m^2 - 2 = 0.$$

This tells us that either $m = \sqrt{2}$ or $m = -\sqrt{2}$. In the initial statement of the problem everything on the right-hand side of the equation was positive, so m must be positive. Hence, we discard the negative solution, and we get $m = \sqrt{2}$. Now let us look at a very similar problem.

The Problem

Find the value of x, where,

$$x = \cfrac{2}{3 - \cfrac{2}{3 - \cfrac{2}{3 - \cfrac{2}{3 - \dots}}}}.$$

Keywords: Infinite series, patterns, recurrence, algebra, quadratics.

First Attempt

Proceeding as before, we substitute the value of x in the right-hand side of the equation. Doing so we get

$$x = \frac{2}{3-x}, \quad \therefore x^2 - 3x + 2 = 0.$$

This tells us that either $x = 1$ or $x = 2$. There is no way to discard either of the solutions. So, we must take a different approach to solve the problem.

A Closer Look

Let us take a step back and think about the problem. What do the three dots at the end signify? What does it mean to find the 'value' of a continued fraction? Well, clearly the three dots at the end says that the pattern continues indefinitely. So, a continued fraction requires infinitely many mathematical symbols to be expressed.

In mathematics, the value of any expression which is not in "closed form" (an expression having finitely many mathematical symbols) is defined as follows: Define a sequence where each term is the expression after being chopped off at finitely many mathematical symbols, successive terms having an increasing number of mathematical symbols. The value that this sequence "approaches" (limit) is defined to be the value of the expression.

For example, the expression 0.1666… (which has infinitely many 6's) is defined as the limit of the sequence:

$$0, \quad 0.1, \quad 0.16, \quad 0.166, \quad 0.1666, \quad 0.16666, \quad \ldots$$

It is easy to see that this sequence approaches the value 1/6. So, we define $0.1666\ldots = 1/6$.

The value of an infinite continued fraction can be defined similarly, as the limit of the sequence formed by chopping the infinite continued fraction at regular intervals and evaluating the finite continued fractions. Any infinite continued fraction of the form

$$a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \cdots}}}$$

can be chopped at regular intervals into finite continued fractions in two ways. This gives two sequences:

$$a_0, \quad a_0 + \frac{b_1}{a_1}, \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2}}, \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3}}}, \quad \ldots$$

and

$$a_0 + b_1, \quad a_0 + \frac{b_1}{a_1 + b_2}, \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + b_3}}, \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_4 + \cdots}}}, \quad \ldots$$

A Different Approach

Applying the same to the proposed problem, we get two sequences:

$$0, \quad \frac{2}{3}, \quad \frac{2}{3-\frac{2}{3}}, \quad \frac{2}{3-\frac{2}{3-\frac{2}{3}}}, \quad \ldots$$

and:

$$2, \quad \frac{2}{3-2}, \quad \frac{2}{3-\frac{2}{3-2}}, \quad \frac{2}{3-\frac{2}{3-\frac{2}{3-2}}}, \quad \ldots$$

The sequences can be simplified as:

$$0, \quad \frac{2}{3}, \quad \frac{6}{7}, \quad \frac{14}{15}, \quad \ldots$$

and

$$2, \quad 2, \quad 2, \quad 2, \quad \ldots$$

The first sequence clearly approaches 1, and the second sequence approaches 2. And we are stuck again. Can this problem be said to have a definite answer at all?

A sequence can be expressed as $\{t_n\}$, where t_n represents the n-th term of the sequence. Notice that, due to the repeating structure, both the sequences formed will follow a single (recurrence) relation expressing the $(n+1)$-th term of the sequence in terms of the previous terms.

Let $\{t_n\}$ be a sequence such that,

$$t_{n+1} = \frac{2}{3 - t_n}.$$

We observe that $\{t_n\}$ describes the first sequence for $t_1 = 0$, and $\{t_n\}$ describes the second sequence for $t_1 = 2$.

Now, in some sense, the continued fraction is not just the two sequences formed; it represents this general recurrence relation. We can calculate a general expression for $\{t_n\}$ in terms of n and t_1 by solving the recurrence relation, after which we can evaluate the limit of $\{t_n\}$ for some general first t_1.

Solving the recurrence relation is slightly involved; it requires some concepts from the field of discrete mathematics (the 'characteristic equation') and is outside the scope of our discussion. After solving the recurrence relation, we get

$$t_n = \frac{2^n (2 - t_1) + 4 (t_1 - 1)}{2^n (2 - t_1) + 2 (t_1 - 1)}.$$

It is not too difficult to verify that this t_n satisfies the recurrence relation.

Now we evaluate the limit of $\{t_n\}$ as n goes to infinity. For all $t_1 \neq 2$, as n becomes large, the 2^n term dominates the constant term and the sequence $\{t_n\}$ approaches the value 1. For $t_1 = 2$, the sequence $\{t_n\}$ approaches the value 2.

We observe that, except a single "pole" at $t_1 = 2$, the recurrence relation describes a sequence which approaches 1 for all $t_1 \neq 2$. So, in our proposed problem we can assign $x = 1$ and discard the value $x = 2$.

Conclusion

A similar approach can be taken to find the value of any infinite continued fraction with repeating structure or one in which a recurrence relation can be established.

For example, we can solve the infinite continued fraction which I mentioned in the introduction using this approach. Again, from the repeating structure we observe that its two sequences formed by chopping the infinite continued fraction at regular intervals also follow a single recurrence relation,

$$c_{n+1} = 1 + \frac{1}{1 + c_n}.$$

The first and second sequences are obtained for $c_1 = 1$ and $c_1 = 2$ respectively. After finding the general expression for $\{c_n\}$ and evaluating the limit of $\{c_n\}$ as n goes to infinity, we find that for all $c_1 \neq -\sqrt{2}$, the sequence $\{c_n\}$ approaches the value $\sqrt{2}$. For $c_1 = -\sqrt{2}$, the sequence $\{c_n\}$ approaches the value $-\sqrt{2}$.

We observe that, except a single "pole" at $c_1 = -\sqrt{2}$, the recurrence relation describes a sequence which approaches $\sqrt{2}$ for all $c_1 \neq -\sqrt{2}$. So, in the problem we can assign $m = \sqrt{2}$ and discard the value $m = -\sqrt{2}$.

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