Zeller's Congruence

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Historical Background

Zeller's Congruence is a method designed to find the day of the week corresponding to any given date in the Gregorian calendar. It was first discovered and proposed by Julius Christian Johannes Zeller, a German mathematician, and published in the journal of the Societe Mathematique de France.

Features of the Gregorian Calendar

In order to understand how exactly this method works, we must recognize which calendar we are applying it to and understand how the Gregorian calendar is designed. The Gregorian Calendar, introduced in 1582 by Pope Gregory as a modification of the Julian Calendar, is the universally accepted version. In this, one earth year consists not of 365.25 but 365.2425 days. To account for the decimal places, we add an additional day to the end of February every four years, since $0.25 \times 4 = 1$. Each such year is called a leap year and it has 366 days.

However, since $365.25 - 365.2425 = 0.0075$, adding one day in February once every four years will produce an over-estimate of $0.0075 \times 400 = 3$ days every 400 years. Thus, it was decided that the centurial years (years that mark the beginning of a century, such as 1800, 2000, etc.) shall be considered leap years only if they occurred every 400 years, that is, if they were divisible by 400. In any period of four consecutive centuries, there will be 24 leap years in the first three centuries and 25 in the fourth century (since the centurial year of that century is a leap year). Therefore, there will be $24 \times 3 + 25 = 97$ leap years for every four centuries in the Gregorian calendar.

Keywords: Calendars, days of week, pattern, cyclic property, modular arithmetic.

Since there are 97 leap years every four centuries, there is a total of $(365 \times 303) + (366 \times 97) = 146097$ days in four centuries. Since $146097 = 20871 \times 7$, there are 20871 weeks in four hundred years, and the next four centuries repeat a similar pattern. We call this the *cyclic property* of the Gregorian Calendar.

Cyclicity is a foundational feature of modular arithmetic. We will be looking at the use of modular arithmetic in finding the days of the week of any given date. This is also the reason why the formula is represented as a congruence and not a fixated equation.

We shall also be describing the *Doomsday Algorithm*, on the fiftieth anniversary of its development by British mathematician John Conway.

Finding the day of the week of any given date

This method consists of finding the first day of the year in which the required date is located and using it as a relation to the day of the week for our required date.

The first day of any given year

Assume for the sake of convenience that the year begins on March 1st, and the week starts with Monday $= 1$.

We first define a few variables. Suppose that:

- k represents the day of the month
- d_N represents the first day of March of a year N
- m represents the number of the month. Here, we have March = 1, April = 2, May = 3, and so on until January = 11 and February = 12.
- C represents the hundreds part of the year and Y represents the tens and units part of the year. For example, if $N = 1776$, then $C = 17$ and $Y = 76$.

The formula for d_N, where N=100C+Y, we have

$$d_N = \left(3 + 5C + Y + \left\lfloor \frac{C}{4} \right\rfloor + \left\lfloor \frac{Y}{4} \right\rfloor\right) \textit{mod}\, 7$$

where the floor $\lfloor x \rfloor$ of a real number x is the greatest integer less than or equal to it.

For example, applying the formula to 1776 gives

$$3 + (17 \times 5) + 76 + \left\lfloor \frac{17}{4} \right\rfloor + \left\lfloor \frac{76}{4} \right\rfloor = 3 + 85 + 76 + 4 + 19 = 187 \equiv 5\, \textit{mod}\, 7.$$

Therefore, March 1, 1776, was a Friday.

Now that we know how to find the day of the week corresponding to March 1st of any year, we can find the first day of an arbitrary month in any given year.

At the beginning of month m, the day of the week shifts $\lfloor 2.6m - 0.2 \rfloor$ from March 1st of that year. This is represented by the formula $d_N + \lfloor 2.6m - 0.2 \rfloor - 2 \bmod 7$.

For example, now that we know the day of the week of 1 March 1776, we can find the day of the week of 1 April 1776 using our formula:

$$d_{1776} + \lfloor 5.2 - 0.2 \rfloor - 2 = 5 + 5 - 2 = 8 \equiv 1 \bmod 7.$$

Thus, April 1, 1776 was a Monday.

Finding the day of the week of an arbitrary date

We shall use the first day of the given year to find the day of the week of our required date.

Let DW represent the required day of the week for an arbitrary date in the format DD/MM/YYYY. Let k be the day of the month in the year $N = 100C + Y$. Then, we have

$$DW = (k-1) + d_N + \lfloor 2.6m - 0.2 \rfloor - 2.$$

Substituting the formula for d_N gives

$$\begin{aligned} DW &= (k-1) + d_N + \lfloor 2.6m - 0.2 \rfloor - 2 \\ &= (k-1) + \left(3 + 5C + Y + \left\lfloor \frac{C}{4} \right\rfloor + \left\lfloor \frac{Y}{4} \right\rfloor\right) + \lfloor 2.6m - 0.2 \rfloor - 2 \\ &= \left(k + 5C + Y + \left\lfloor \frac{C}{4} \right\rfloor + \left\lfloor \frac{Y}{4} \right\rfloor + \lfloor 2.6m - 0.2 \rfloor\right) \pmod 7. \end{aligned}$$

Example 1: We shall find the day of the week of the birthdate of the prominent Indian mathematician Srinivasa Ramanujan, which is 22nd December 1887.

We have $k = 22$, $m = 10$, $C = 18$, $Y = 87$. Applying our formula, we have:

$$\begin{aligned} DW &= \left(22 + (5 \times 18) + 87 + \left\lfloor \frac{18}{4} \right\rfloor + \left\lfloor \frac{87}{4} \right\rfloor + \lfloor 2.6 \times 10 - 0.2 \rfloor\right) \pmod 7 \\ &= (22 + 90 + 87 + 4 + 21 + 25) \pmod 7 \\ &= 249 \pmod 7 \equiv 4 \pmod 7. \end{aligned}$$

Therefore, Srinivasa Ramanujan was born on a Thursday.

Example 2: Finding the day of the week on which India gained independence (15th August 1947).

We have $k = 15$, $m = 6$, $C = 19$, $Y = 47$. We apply our formula to get

$$\begin{aligned} DW &= \left(15 + (5 \times 19) + 47 + \left\lfloor \frac{19}{4} \right\rfloor + \left\lfloor \frac{47}{4} \right\rfloor + \lfloor 2.6 \times 6 - 0.2 \rfloor\right) \pmod 7 \\ &= 15 + 95 + 47 + 4 + 11 + 15 \pmod 7 \\ &= 187 \pmod 7 \equiv 5 \pmod 7. \end{aligned}$$

Therefore, India gained independence on a Friday.

The Doomsday Algorithm (1973)

We close by describing John Conway's Doomsday Algorithm. Like Zeller's Congruence, it uses the periodic nature of the Gregorian calendar.

This algorithm is based on the observation that, in a year, certain dates fall on the same day of the week (trivially so if the gap between the two dates is an exact number of weeks, with no leap year complication). This verification may be done mentally without using complicated expressions and variables. This makes use of a 'landmark' date, called *Doomsday*, and knowing the day of the week of the Doomsday helps us find the day of the week of other dates in that year.

In order to find the day of the week of a given day, we need to know the Doomsday or the 'anchor day' of that year. Note that Doomsday 1900 was Wednesday, and that Doomsday increases by 1 every 12 years. The algorithm developed by Conway to find Doomsday of any given year in the 20th century is as follows:

(1) Find the quotient Q_1 when the last two digits of the year are divided by 12.

(2) Find the remainder R of the division in Step 1.

(3) Find the quotient Q_2 when this remainder (R) is divided by 4

(4) Add these three numbers to the Doomsday 1900 (Wednesday $= 3$) and take the final number modulo 7.

Consider odd-numbered and even-numbered months. Odd numbered months are January, March, May, July, September, and November, while the even numbered months are February, April, June, August, October, and December.

For even months, the n-th day of the nth numbered even month is Doomsday. That is, February 2nd, April 4th, June 6th, and so on until 12th December are all Doomsdays.

For odd-numbered months, the 9th day of the 5th month, the 5th day of the 9th month, the 11th day of the 7th month and the 7th day of the 11th month are all Doomsdays. (This can be memorized using the mnemonic "I work from 9 am to 5 pm at the 7-11", as knowing the above information requires the use of these 4 numbers in a mnemonic.)

Example: What day of the week was John Conway's birthdate?

John Conway was born on 26th December 1937. Applying the Doomsday Algorithm to 1937, we get the following. Dividing 37 by 12 gives quotient 3 and remainder 1, while dividing 1 by 4 gives a quotient of 0. Adding 3, 1 and 0 to Wednesday (Doomsday 1900, numbered 3) gives $3 + 1 + 0 + 3 = 7$, which taken modulo 7 is 0. Therefore, Doomsday 1937 is a Sunday.

According to our even month observations, 12th December 1937 was also Doomsday, and since the 26th December falls exactly two weeks from this date, it was a Sunday.

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