Problem Corner

Congruency, A Trigonometric View

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In the article "Another Theorem for Congruence of Triangles" by Kasi Rao Jagathapu [1], published in *At Right Angles*, March 2023, the author points to the need for formulating additional congruence theorems. (See also [2].) It may be useful to look at this question from a trigonometric point of view.

Suppose we are given two sides of a triangle and one of the angles. Under what circumstances do these parameters uniquely fix the triangle? We consider the different possibilities.

Let the triangle be labelled ABC, and let the given sides be b and c. To fix the discussion, we assume that $b > c$. (We consider the much simpler case $b = c$ later.)

Suppose the angle specified is $\measuredangle A$ (the included angle). In this case we can calculate the third side a via the cosine rule ($a^2 = b^2 + c^2 - 2bc\cos A$). With three sides specified, the triangle is uniquely fixed. This obviously corresponds to the side-angle-side (SAS) congruence theorem.

Next, suppose the angle specified is $\measuredangle B$ (the non-included angle opposite the *longer* side). Since $c < b$, it follows that $\measuredangle C < \measuredangle B$ (strictly). Using the sine rule, we compute the value of $\sin C$:

$$\frac{\sin B}{b} = \frac{\sin C}{c}, \quad \therefore \ \sin C = \frac{c \sin B}{b}.$$

Knowing $\sin C$, we can determine a pair of supplementary angles whose sine is this value (recall that θ and $180^\circ - \theta$ have equal sines). One of these angles is obtuse and the other is acute. Angle C cannot be the obtuse angle, since $\measuredangle C < \measuredangle B$. *Therefore $\measuredangle C$ must be the acute angle*, which means that it is known. As we know both $\measuredangle B$ and $\measuredangle C$, we also know $\measuredangle A$, and therefore also side a; so the triangle is uniquely fixed.

Keywords: Congruency, side-angle-side (SAS) congruence, cosine rule, sine rule

To give a numerical example, suppose that $b = 5$, $c = 4$, and $\measuredangle B = 65°$. Then we have, from the above relationships:

$$\sin C = \frac{4 \cdot \sin 65°}{5} \approx 0.725,$$

so $\measuredangle C \approx 46.47°$ **or** $133.53°$. The latter option is not possible because we must have $\measuredangle C < \measuredangle B$; hence $\measuredangle C \approx 46.47°$. Therefore $\measuredangle A \approx 68.53°$, and the triangle is now determined fully. The situation has been sketched in Figure 1.

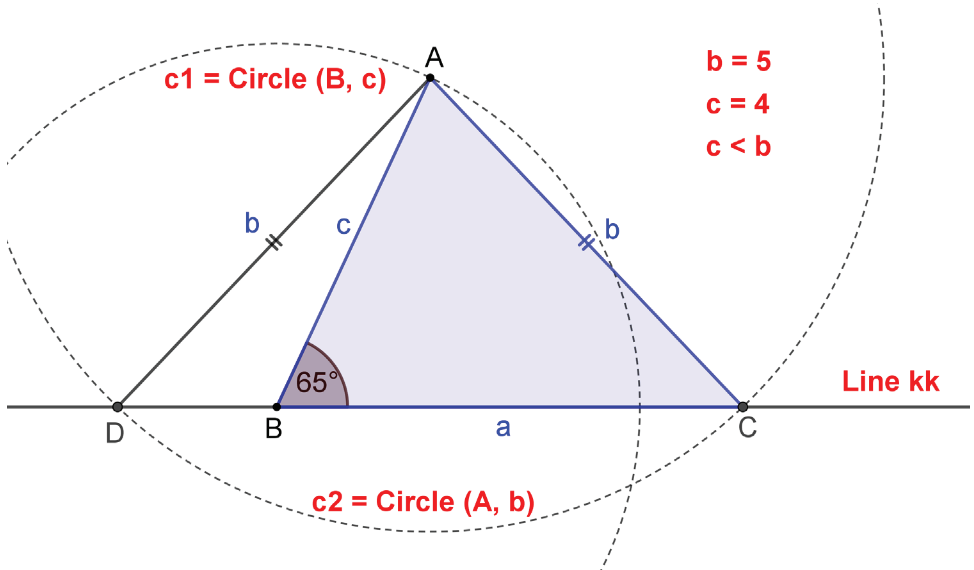

Figure 1. Construction of $\triangle ABC$ given b, c, $\measuredangle B$, with $b > c$

The steps of the construction are these:

1. Draw any arbitrary line kk and mark a point B on kk.
2. Draw a ray through B at the given angle $\measuredangle B$ with kk.
3. Draw a circle $c1$ with centre B and radius c. Let the ray through B meet $c1$ at A. This defines vertex A of the required triangle.
4. Draw a circle $c2$ with centre A and radius b. Since $b > c$, circle $c2$ will meet line kk at two points C and D, one on either side of B.
5. Let C be the point such that $\measuredangle ABC$ is equal to the given angle $\measuredangle B$. This defines vertex C of the required triangle, which is now fully determined.

It follows that if we know two sides of a triangle as well as the angle opposite the *longer* of the two sides, the triangle is uniquely fixed. This corresponds to the new congruency theorem proved in [1].

Now suppose the angle specified is $\measuredangle C$ (the non-included angle opposite the *shorter* side). As earlier, we are able to deduce the value of $\sin B$ via the sine rule, and thus restrict $\measuredangle B$ to two possibilities (a pair of supplementary angles). *But the logic used earlier to eliminate one of the two possibilities no longer applies*; we are not able to uniquely fix $\measuredangle B$. So we do not get any result here, and we do not obtain a congruence theorem.

To give a numerical example, suppose that $b = 4$, $c = 5$, and $\measuredangle B = 45^\circ$. Then we have, from the above relationships:

$$\sin C = \frac{5 \cdot \sin 45^\circ}{4} \approx 0.884,$$

so $\measuredangle C \approx 62.11^\circ$ **or** 117.89°. Now both options must be considered; neither one can be discarded. Accordingly, there are two possible triangles ($\triangle ABC$ and $\triangle ABC'$) with these specifications. See Figure 2. (The construction steps are the same as earlier.)

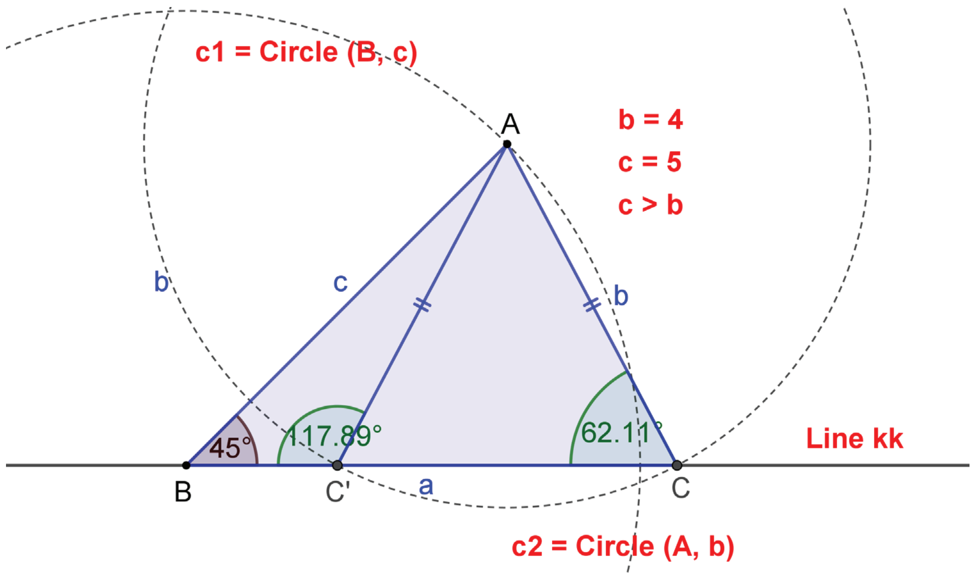

Figure 2. Construction of $\triangle ABC$ given b, c, $\measuredangle B$, with $c > b$

We round off the discussion by considering the case when the given sides are equal to each other. That is, we must construct triangle ABC given the sides b, c, with $b = c$, and the angle B. This is straightforward. For we have $\measuredangle B = \measuredangle C$ (since $b = c$), so $\measuredangle A - 180^\circ - 2\measuredangle B$ is known, and therefore also side a, via either the sine rule or the cosine rule. Thus the triangle is fixed uniquely. Here too we obtain a congruency theorem.

Summarising, we may say that if we are required to construct triangle ABC given the sides b and c where $b \geq c$, and we are also given (the non-included) $\measuredangle B$, then the triangle is fixed uniquely.

References

1. Kasi Rao Jagathapu, "Another Theorem for Congruence of Triangles," *At Right Angles*, March 2023; from https://publications.azimpremjiuniversity.edu.in/4548/

2. A. Ramachandran, "Congruency and constructibility in triangles," *At Right Angles*, March 2017; from https://publications.azimpremjiuniversity.edu.in/1366/

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