Two 4-Digit Puzzles

ANAND PRAKASH

In this short article we pose and solve two playful problems about 4-digit numbers.

Notation. By $\boxed{abcd}$ we mean the 4-digit number whose digits are, respectively, a, b, c, d. So

$$\boxed{abcd} = 10^3 a + 10^2 b + 10c + d. \qquad (1)$$

Similarly for 3-digit numbers and 2-digit numbers; the meaning should be clear.

Problem 1. Find all 4-digit numbers $\boxed{abcd}$ with the property

$$a^2 + b^2 + c^2 + d^2 = \boxed{ab} + \boxed{cd}. \qquad (2)$$

Problem 2. Find all 4-digit numbers $\boxed{abcd}$ with the property

$$a^2 + b^2 + c^2 + d^2 = \boxed{abc} + d. \qquad (3)$$

Solution to Problem 1. Equation (2) may be rewritten as $a^2 + b^2 + c^2 + d^2 = 10a + b + 10c + d$, i.e., as

$$b(b-1) + d(d-1) = a(10-a) + c(10-a). \qquad (4)$$

Observe that (4) is symmetric in $\{a, c\}$ and also symmetric in $\{b, d\}$. Hence we may as well impose the additional restrictions $a \le c$ and $b \le d$. No essential loss results from these restrictions.

Since $b, d \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, it follows that $b(b-1),\ d(d-1) \in \{0, 2, 6, 12, 20, 30, 42, 56, 72\}$. Hence $b(b-1) + d(d-1)$ can assume only the values shown in Table 1.

Similarly, since $a, c \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, it follows that $a(10-a),\ c(10-c) \in \{9, 16, 21, 24, 25\}$. Hence $a(10-a) + c(10-c)$ can assume only the values shown in Table 2.

Keywords: Place value, constraints

0	2	4	6	8	12	14	18
20	22	24	26	30	32	36	40
42	44	48	50	54	56	58	60
62	68	72	74	76	78	84	86
92	98	102	112	114	128	144	

Table 1. Possible values of $b(b-1)+d(d-1)$ for $b,d \in \{0,1,2,\ldots,8,9\}$

18	25	30	32	33
34	37	40	41	42
45	46	48	49	50

Table 2. Possible values of $a(10-a)+c(10-c)$ for $a,c \in \{0,1,2,\ldots,8,9\}$

The numbers common to the two collections are the following:

$$18,\ 30,\ 32,\ 40,\ 42,\ 48,\ 50. \tag{5}$$

So to solve the stated problem, we must look for values of a, b, c, d such that

$$b(b-1)+d(d-1)=k=a(10-a)+c(10-c), \quad \text{where } k \in \{18,30,32,40,42,48,50\}. \tag{6}$$

We take up each k-value in turn.

- $\boxed{k=18}$
  We clearly have $(a,c)=(1,1)$ and $(b,d)=(3,4)$. Hence $\boxed{abcd}=1314$. (Of course, 1413 is also a solution but we choose not to list it.)
- $\boxed{k=30}$
  Here we have $a,c=(1,3)$ and $(b,d)=(1,6)$. Hence $\boxed{abcd}=1136$. (We do not list the permutations of this which are also solutions: 3116, 1631, 3611.)
- $\boxed{k=32}$
  Here we have $a,c \in \{2,8\}$ and $(b,d)=(2,6)$. Hence $\boxed{abcd}=2226, 2286$, or 8286. (As earlier, we do not list other solutions that are permutations of these.)
- $\boxed{k=40}$
  Here we have $a,c=(2,4)$ and $(b,d)=(5,5)$. Hence $\boxed{abcd}=2545$. (As earlier, we do not list other solutions that are permutations of these.)
- $\boxed{k=42}$
  Here we have $a,c \in \{3,7\}$ and $(b,d)=(1,7)$. Hence $\boxed{abcd}=3137$ or 3177.
- $\boxed{k=48}$
  Here we have $a,c \in \{4,6\}$ and $(b,d)=(3,7)$. Hence $\boxed{abcd}=4347$ or 4367.

- $\boxed{k = 50}$

  Here we have $a, c = (5,5)$ and $(b,d) = (5,6)$. Hence $\boxed{abcd} = 5556$. (As earlier, we do not list other solutions that are permutations of these.)

Hence the set of solutions is the following:

$$1314,\ 1136,\ 2226,\ 2286,\ 8286,\ 2545,\ 3137,\ 3177,\ 4347,\ 4367,\ 5556. \qquad (7)$$

We have chosen not to list permutations of these solutions where $a > c$ or $b > d$ (though they may satisfy the stated condition). □

Solution to Problem 2. Equation (3) may be rewritten as $a^2 + b^2 + c^2 + d^2 = 100a + 10b + c + d$, i.e., as

$$a(100 - a) + b(10 - b) = c(c - 1) + d(d - 1). \qquad (8)$$

The set of all numbers of the type $c(c-1) + d(d-1)$ for $c, d \in \{0, 1, 2, \ldots, 8, 9\}$ has been listed earlier, in Table 1. For convenience, we have listed the values again (Table 3).

The set of all numbers of the type $a(100 - a) + b(10 - b)$ for $a, b \in \{0, 1, 2, \ldots, 8, 9\}$ may be similarly computed; we have listed the values in Table 4.

It takes just a moment's glance to verify that this there are no numbers in common between the two collections.

It follows that Problem 2 has no solutions. □

0	2	4	6	8	12	14	18
20	22	24	26	30	32	36	40
42	44	48	50	54	56	58	60
62	68	72	74	76	78	84	86
92	98	102	112	114	128	144	

Table 3. Possible values of $c(c-1) + d(d-1)$ for $c, d \in \{0, 1, 2, \ldots, 8, 9\}$

108	115	120	123	124	205	212	217	220	221
300	307	312	315	316	393	400	405	408	409
484	491	496	499	500	573	580	585	588	589
660	667	672	675	676	745	752	757	760	761
828	835	840	843	844					

Table 4. Possible values of $a(100 - a) + b(10 - b)$ for $a, b \in \{0, 1, 2, \ldots, 8, 9\}$

ANAND PRAKASH runs a small garment shop at Kesariya village in the state of Bihar. He has a keen interest in number theory and recreational mathematics and has published many papers in international journals in these fields. He also has a deep interest in classical Indian music, poetry, and cooking. He has written a large number of poems in Hindi. He may be contacted at prakashanand805@gmail.com.