Irrational Nine-Point Centre is Impossible for a Triangle with Rational Vertices

SIDDHARTHA SANKAR CHATTOPADHYAY

In this short note, we prove that an irrational point in the Euclidean plane cannot be realized as the nine-point centre of a triangle all of whose vertices are rational points.

In the Iranian Mathematics competition at The University of Isfahan in March 1978, the following problem was given.

Problem 1. [1, 1.6.1] In the *xy*-plane a point is called 'rational' if both of its coordinates are rational.

Prove that if the centre of a given circle in the plane is not rational, then there are at most two rational points on the circle.

For the sake of convenience, we call a point 'irrational' if it is not rational as per the definition in Problem 1. The idea to solve Problem 1 is to assume, for the sake of contradiction, that there are three rational points on a circle whose centre is irrational and then arrive at a contradiction. Therefore, we can reformulate the above problem and assert that the circumcentre of a triangle is rational if the vertices of the triangle are all rational. In this article, we prove an analogous result for the centre of the nine-point circle, often referred to as the 'nine-point centre', for a triangle with rational vertices. The precise statement is as follows.

Keywords: Circumcircle, circumcenter, nine-point circle, nine-point centre, rational number, irrational number, rational point, irrational point

Theorem 1. Let P be an irrational point in the Euclidean plane. Then there does not exist any triangle ABC with all of A, B and C being rational points, such that P is the nine-point centre of △ABC.

Before we proceed to prove Theorem 1, we recall that the nine-point circle of $\triangle ABC$ is the circle passing through the midpoints of the sides, the feet of perpendiculars drawn from the vertices to their respective opposite sides, the midpoints of the line segments joining the vertices to the orthocentre. Also, the nine-point centre is the midpoint of the line segment joining the circumcentre and the orthocentre of $\triangle ABC$.

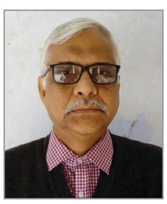
Proof of Theorem 1. Let $\triangle ABC$ be a triangle in the Euclidean plane such that A, B and C are all rational. Let the co-ordinates of A, B and C be (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively, where x_1, x_2, x_3, y_1, y_2 and y_3 are rational numbers.

We call a straight line $Ux + Vy = W$ 'rational' if U, V and W are rational numbers. Since B and C are rational points, the straight line joining these two points is rational. Also, the midpoint D of the line segment BC is a rational point because the co-ordinates are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$. Therefore, the perpendicular bisector of BC is a rational straight line. Similarly, the perpendicular bisector of AC is rational and thus the point of intersection of these two straight lines, i.e., the circumcentre of $\triangle ABC$, is a rational point.

Using the fact that BC is a rational straight line and A is a rational point, by a similar argument as above, we conclude that the straight line passing through A and perpendicular to BC is rational. Such is the case for the other two feet of perpendiculars drawn from the vertices B and C. Consequently, their point of intersection is a rational point. In other words, the orthocentre of $\triangle ABC$ is also a rational point. Therefore, the nine-point centre, which is the midpoint of the straight line segment joining the orthocentre and the circumcentre, is a rational point. □

References:

[1] B. R. Yahaghi, *Iranian Mathematics Competitions 1973–2007*, Hindustan Book Agency.



SIDDHARTHA SANKAR CHATTOPADHYAY is a retired Mathematics teacher from Bidhannagar Govt. High School, Salt Lake, Kolkata. He is actively associated with the Kolkata based 'Association for Improvement of Mathematics Teaching' for 30 years. He has participated in seminars and workshops for school students organized by Jagadis Bose National Science Talent Search over the past 15 years as a resource person, and has been engaged in spreading Mathematics Education across West Bengal by organizing seminars and workshops. He may be contacted at 1959ssc@gmail.com