एकवृन्तगतफलद्वयन्यायः।
'Two Fruits on One Stalk'

A S RAJAGOPALAN

एकवृन्तगतफलद्वयन्यायः is a popular maxim in Sanskrit literature used to describe situations when one effort gives two or many results simultaneously. Sometimes while solving problems, we stumble upon interesting results different from the intended results. One such example is given below.

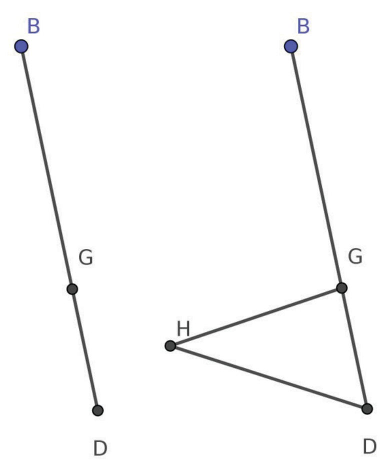
Problem: Using ruler and compass, construct the triangle whose median lengths are given.

Construction method

Let the given median lengths be represented by Ma (from vertex A), Mb (from vertex B), and Mc (from vertex C).

Step 1. Draw BD = Mb, and mark point G on BD at a length of $\frac{1}{3}$Mb from D.

Step 2. Construct ΔGDH, with HG = $\frac{1}{3}$Ma and HD = $\frac{1}{3}$Mc. (Use standard construction methods to trisect segments Ma, Mb, and Mc.)



Step 1 Step 2

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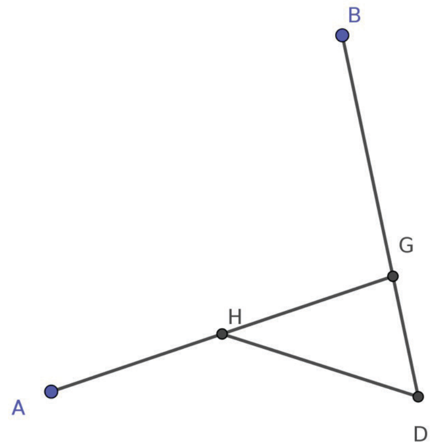
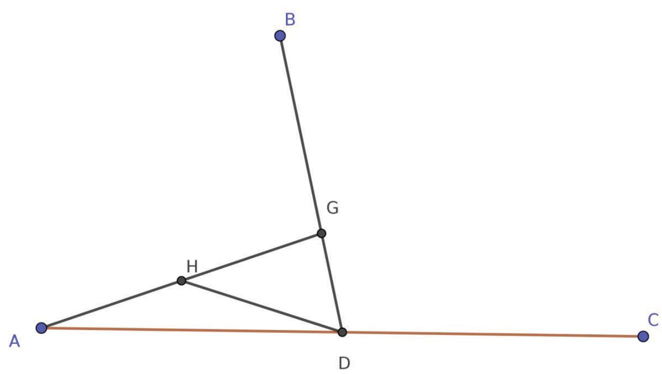
Step 3. Extend GH to point A with GH = HA.



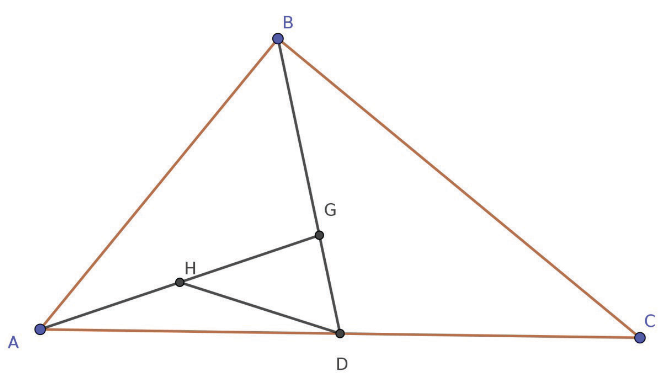
Step 3

Step 4. Construct segment AD and extend AD to point C with AD = DC.



Step 4

Step 5. Construct ΔABC. This triangle will have median lengths as Ma, Mb, and Mc.



Step 5

Analysis and justification

- Refer Figure 1.
- Let BD = Mb, AE = Ma, and CF = Mc be the medians of ΔABC to be constructed using compass and ruler. G is the centroid of ΔABC which divides the medians in the ratio of 2 : 1, the larger parts being towards the vertices of the triangle.
- Let H be the mid-point of AG. Since AG is $\frac{2}{3}$Ma, HG = $\frac{1}{3}$Ma.
- In ΔAGC, H and D being midpoints of sides AG and AC, HD will be half of GC and parallel to GC (Midpoint theorem).
- Since GC = $\frac{2}{3}$Mc and HD being half of GC, HD = $\frac{1}{3}$Mc.
- Now HD = $\frac{1}{3}$Mc, HG = $\frac{1}{3}$Ma and GD = $\frac{1}{3}$Mb.
- We started the construction with median BD and ΔGDH with HG = $\frac{1}{3}$Ma, HD = $\frac{1}{3}$Mc, and GD = $\frac{1}{3}$Mb.
- GH was extended to get point A (HA = GH) and AD was extended to get point C (AD = DC), thus completing the construction of ΔABC.

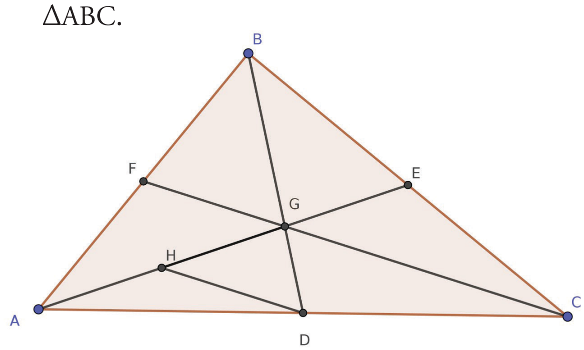


Figure 1

Note: Another interesting fact can be inferred from the above figure. ΔGDH has HG = $\frac{1}{3}$Ma, HD = $\frac{1}{3}$Mc and GD = $\frac{1}{3}$Mb. Therefore, ΔGDH is a scaled-down (similar triangle) version of the triangle formed by Ma, Mb, and Mc (medians of ΔABC) with a scale factor of 3 : 1.

If we take the area of ΔGDH to be 1 sq. unit, then the area of the triangle formed by Ma, Mb, and Mc (medians of ΔABC) will be 9 sq. units since these two triangles are similar with a scale factor of 3 : 1.

Since HG = HA, Area of ΔGDH = Area of ΔADH = 1 sq. unit (equal base and same height).

Area of ΔADG = Area of ΔGDH + Area of ΔADH = 2 sq. units.

Since AD = DC, Area of ΔADG = Area of ΔCDG = 2 sq. units (equal base and same height).

Area of ΔAGC = Area of ΔADG + Area of ΔCDG = 4 sq. units.

But Area of ΔAGC = $\frac{1}{3}$Area of ΔABC. Hence area of ΔABC = 3 times area of ΔAGC = 12 sq. units.

From the above it can be seen that area of the triangle formed by M*a*, M*b*, and M*c* (medians of ΔABC) is 9 sq. units and area of ΔABC is 12 sq. units.

Therefore, the area of the triangle formed by the medians of a triangle is $\frac{3}{4}$ the area of the parent triangle.

A S RAJAGOPALAN has been teaching in Rishi Valley School KFI for the past 18 years. He teaches Mathematics as well as Sanskrit. Earlier, he was working as an engineer. He is keenly interested in teaching mathematics in an engaging way. He has a deep interest in classical Sanskrit literature. He enjoys long-distance running. He may be contacted at ayilamraj@gmail.com