ClassRoom

On a Link between Three Trigonometric Identities for a Triangle

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For a ΔABC we have the following three important and powerful trigonometric identities:

I_1. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

I_2. $c = a\cos B + b\cos A$, and similar identities for a and b.

I_3. $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$, and similar identities for $\cos A$ and $\cos B$.

In this short note we show that any one of these results can be used to prove the other two (using only standard trigonometric manipulations and without the need for any diagram).

Proof:

Let $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{1}{k}$, where k represents a non-zero constant.

Keywords: sine rule, cosine rule, proofs, connections

To prove $I_1 \implies I_2$:

We have,

$$\begin{aligned} c &= k \sin C \\ &= k \sin(A+B) \\ &= k[\sin A.\cos B + \cos A.\sin B] \\ &= (k\sin A).\cos B + (k \sin B)\cos A \\ &= a\cos B + b\cos A. \end{aligned}$$

To prove $I_1 \implies I_3$:

From $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{k}$ we get:

$$\begin{aligned} &a^2+b^2-c^2 \\ &= k^2\sin^2 A + k^2 \sin^2 B - k^2 \sin^2 C \\ &= k^2\left(\sin^2 A + \sin^2 B - \sin^2 C\right) \\ &= k^2\left[\sin^2 A + \sin(B+C)\sin(B-C)\right] \\ &\quad \text{(we justify this step in the Appendix)} \\ &= k^2[\sin A \sin(B+C) + \sin A \sin(B-C)] \\ &= k^2 \sin A[\sin(B+C) + \sin(B-C)] \\ &= k^2 \sin A \cdot 2\sin B \cdot \cos C \\ &= 2ab\cos C, \end{aligned}$$

therefore

$$\cos C = \frac{a^2+b^2-c^2}{2ab}.$$

To prove $I_2 \implies I_3$

We have,

$$\begin{aligned} &a^2+b^2-c^2 \\ &= a.a + b.b - c.c \\ &= a(b\cos C + c\cos B) + b(a\cos C + c\cos A) \\ &\quad - c(a\cos B + b\cos A) \\ &= 2ab\cos C \\ &\Rightarrow \cos C = \frac{a^2+b^2-c^2}{2ab} \end{aligned}$$

To prove $I_2 \implies I_1$

We have,

$$\begin{aligned} c &= a\cos B + b\cos A, \\ a &= b\cos C + c\cos B. \end{aligned}$$

Therefore:

$$\begin{aligned} &c^2 - a^2 \\ &= c\cdot c - a\cdot a \\ &= c(a\cos B + b\cos A) - a(b\cos C + c\cos B) \\ &= b(c\cos A - a\cos C) \\ &= (c\cos A + a\cos C)\cdot(c\cos A - a\cos C) \\ &= c^2\cos^2 A - a^2\cos^2 C. \end{aligned}$$

Hence:

$$\begin{aligned} c^2\left(1-\cos^2 A\right) &= a^2\left(1-\cos^2 C\right), \\ \therefore c^2 \sin^2 A &= a^2\sin^2 C, \\ \therefore c\sin A &= a\sin C. \end{aligned}$$

(Since A, B, C are the angles of a triangle, $\sin A$, $\sin B$, $\sin C$ are positive quantities; no negative signs are introduced when we take square roots.) From the last we get

$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

By symmetry it follows that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

To prove $I_3 \implies I_1$

We have,

$$\begin{aligned} &(2bc\sin A)^2 \\ &= 4b^2c^2\left(1-\cos^2 A\right) \\ &= 4b^2c^2\left(1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2\right) \\ &= 4b^2c^2 - (b^2+c^2-a^2)^2 \\ &= (2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2) \\ &= \left\{(b+c)^2 - a^2\right\}\{a^2-(b-c)^2\} \\ &= (a+b+c)(b+c-a)(c+a-b)(a+b-c). \end{aligned}$$

Then is symmetrical in a, b, c

Hence by symmetry $2bc\sin A = 2ac\sin B = 2ab\sin C$, so

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

To prove $I_3 \implies I_2$

We have,

$$\begin{aligned} &a\cos B + b\cos A \\ &\quad = a\cdot\frac{c^2+a^2-b^2}{2ac} + b\cdot\frac{b^2+c^2-a^2}{2bc} \\ &\quad = \frac{1}{2c}\left(c^2+a^2-b^2+b^2+c^2-a^2\right) \\ &\quad = c. \end{aligned}$$

Appendix

$$\begin{aligned} &\sin^2 B - \sin^2 C \\ &\quad = (\sin B + \sin C)(\sin B - \sin C) \\ &\quad = 2\,\sin\frac{B+C}{2}\cos\frac{B-C}{2}\cdot 2\,\sin\frac{B-C}{2}\cos\frac{B+C}{2} \\ &\quad = 2\,\sin\frac{B+C}{2}\cos\frac{B+C}{2}\cdot 2\,\sin\frac{B-C}{2}\cos\frac{B-C}{2} \\ &\quad = \sin(B+C)\sin(B-C)\,. \end{aligned}$$

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