ClassRoom

Nesting Platonic Solids

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What are Platonic Solids?

Platonic solids are a type of regular convex polyhedra, that are made up of a number of regular faces meeting at a vertex. There are five known platonic solids, namely the dodecahedron, cube, tetrahedron, octahedron, and the icosahedron.

What are nested platonic solids?

The idea of nested platonic solids involves perfectly placing or 'fitting' platonic solids within each other. There are multiple combinations to nest the five platonic solids. One of these ways include nesting them in the following order:

- an icosahedron sits inside an octahedron with each of the 12 vertices of the icosahedron touching an edge of the octahedron,
- the octahedron is placed inside the tetrahedron, with each alternate face touching the equilateral triangle in the centre of each face of the tetrahedron,
- the tetrahedron is placed in a cube with four of its vertices touching diagonal vertices of two parallel square bases,
- which, i.e., the cube, finally nests in a dodecahedron with two horizontal diagonal vertices of every pentagon coinciding with the vertices of the cube.

Keywords: Platonic Solids, nesting solids, geometry, congruence, golden ratio.

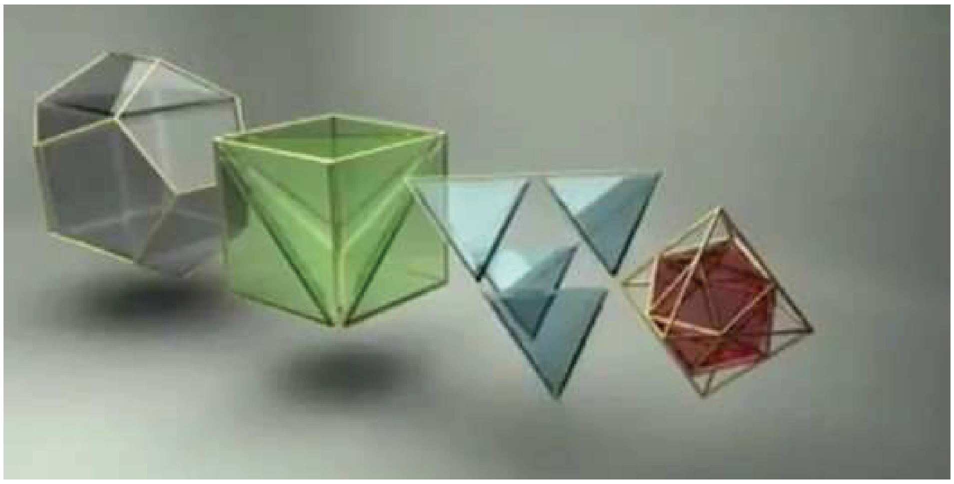

Figure 1. (Source: https://youtu.be/gwxQfujwWrw)

So, what dimensions work to nest the platonic solids in order?

To begin with nesting platonic solids, one needs to decide the dimension of the innermost solid. Intuitively, bigger the dimension of the innermost solid, larger will be the subsequent solids as the ratio between the dimensions of the solids remains fixed in order to nest them in a given way. I started out by constructing an icosahedron of side 4cm. To find dimensions of subsequent nesting solids, I watched this video (https://youtu.be/gwxQfujwWrw) multiple times to fully understand how the icosahedron sits inside the next solid, i.e., the octahedron.

During initial attempts, despite rewatching the video multiple times, some of my doubts about the placement of the icosahedron inside the octahedron remained. So, I decided to make a dummy octahedron of double the dimensions of that of the icosahedron[1]. This helped me understand how the two solids existed, one within the other. It also provided perspective on what the dimensions of the octahedron could be, as I was able to trace the boundary of the octahedron where the top half was supposed to close in order for the icosahedron to sit inside completely without any extra space or gaps.

After observing, understanding and confirming the fact that an icosahedron nested by an octahedron has each of its vertices touching a unique edge of the octahedron, I proceeded to work on closer approximations, considering 2D and 3D orientations.

To find the side of the octahedron, consider the orientation of the icosahedron in the top view of the nesting. The vertices of the icosahedron touch the edge of the octahedron in two ways.

(1) In the first possible way, the two vertices, e.g., G and H, of the icosahedron touch two adjacent edges of the octahedron, in which case the edge, e.g., GH, of the icosahedron spans the distance s between the two contact points on the boundary ('perimeter' of the square base $CD_1E_1F_1$) of the octahedron.

1 Intuitively, the octahedron of double the dimensions will be able to accommodate the chosen icosahedron. Consider the following: if a section of the icosahedron were to lay flat on a plane, then the height of the section in that case will be given by 3 times the median length of the equilateral triangle (building unit of the icosahedron), i.e., $3\left(\frac{\sqrt{3}}{2}\right)4 = 6\left(\sqrt{3}\right)$. This approximation will also give a larger dimension for the enclosing octahedron because it is derived from the assumption that the section lies in the plane. However, it tells us that an octahedron of dimension 8 cm (i.e., twice the side of the icosahedron and greater than the obtained length on the plane) will also be able to fully contain the icosahedron.

(2) In the second way, the vertex (e.g., L) of the icosahedron sits on one of the inner edges (either above or below the base square) of the octahedron.

In the former case, a right-angled isosceles triangle ($\triangle CGH$, right angled at C) is obtained at the corner, as the icosahedron is uniformly seated inside the octahedron.

The latter case (e.g., L) also forms an isosceles right-angled triangle with legs h (e.g., D_1K and D_1G) and hypotenuse d (diagonal of the 2D pentagon, e.g., $LGMNK$, formed on top of the icosahedron), e.g., diagonal GK forms $\triangle GD_1K$, right angled at D_1. The sum of two different legs, i.e., $s + h = D_1G + GC$, from these two right isosceles triangles gives the side of the octahedron D_1C.

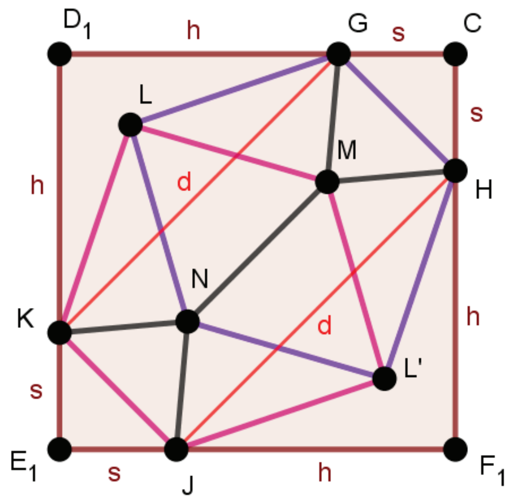

Figure 2. Top view of nesting an icosahedron inside an octahedron - $CD_1E_1F_1$ is a square base of the double pyramid, i.e., octahedron.

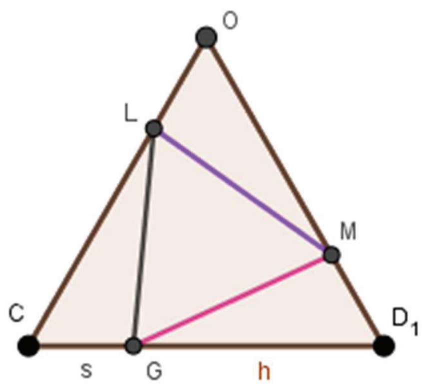

Figure 3. Front view of an icosahedron placed inside an octahedron - icosahedron face overlapping an octahedron face.

When a face of the octahedron with the embedded face of the icosahedron in Figure 3 is considered, the congruence of triangles $\triangle GMD_1$ (PQB), $\triangle LGC$ (RPA) and $\triangle MLO$ (QRO) can be looked at using rotational symmetry. Consider without loss of generality, $\triangle GMD_1$ and $\triangle LGC$. It is known that $GM = GL$ (PQ = PR) (sides of an equilateral triangle). By rotating the equilateral triangles $\triangle CD_1O$ (ABO) and $\triangle GML$ (PQR) 120° clockwise about their common centre it can be concluded without ambiguity that $GD_1 = CL$ (PB = AR) $= h$ and $MD_1 = GC$ (QB = PA) $= s$ as the two triangles superimpose on each other (Figure 3). Therefore, $\triangle GMD_1 \cong \triangle LGC \cong \triangle MLO$, by SSS criterion using rotational symmetry of equilateral triangles.

In Figure 2, the pink and purple line segments show two pentagonal sections of an icosahedron, depicted three-dimensionally.

Length of side of octahedron $= s + h$ where $s = CG$ and $h = GD_1$ from Figure 2.

From the right isosceles $\triangle CGH$,

$$2s^2 = 16 \Rightarrow s^2 = 8 \Rightarrow s = 2\sqrt{2}$$

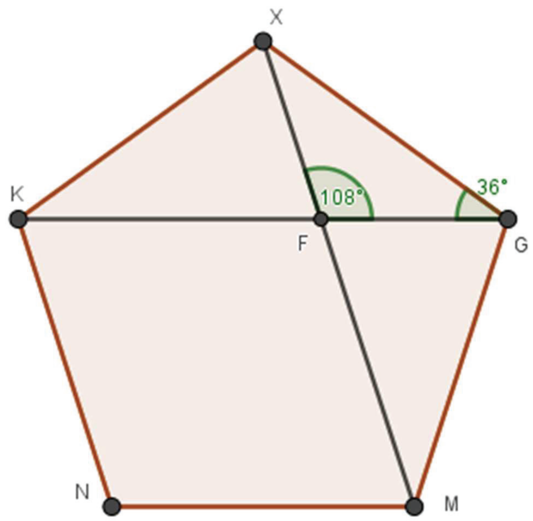

Figure 4

In Figure 4 above, the four points of the pentagon K,N,M,G are obtained using Figure 2, while X is the fifth vertex lying behind and joining L, which is not visible in the two-dimensional Figure 2. So, side of the icosahedron $= KN = KX = GX = 4$.

In Figure 4, the acute angles of $\triangle GFX$ and $\triangle GXK$ are equal (angles adjacent to bases of $\triangle GXK \cong \triangle MGX$). Therefore,

(1) The remaining angles are equal, i.e., $\measuredangle GFX = \measuredangle GXK = 108°$ (internal angle of a regular pentagon)

$\Rightarrow$The acute angles, viz. $\measuredangle XGK = \measuredangle XKG = \measuredangle FXG = \measuredangle FGX = \frac{1}{2} \times \measuredangle XFG = \frac{1}{2}(180° - \measuredangle XFG)$

$= \frac{1}{2}(180° - 108°) = 36°$

$\Rightarrow \measuredangle KXF = 180° - (\measuredangle XFK + \measuredangle XKF) = 180° - (72° + 36°) = 72° = \measuredangle XFK$

$\Rightarrow KF = KX$ (opposite sides of equal angles are equal) $= 4$

(2) $\triangle GFX \sim \triangle GXK$

$\Rightarrow$ by proportional side argument of similar triangles $\frac{FG}{GX} = \frac{GX}{GK}$.

Recall that GK $= d$ from Figure 2. Therefore, $FG = GK - KF = d - 4$. So, $\frac{d-4}{4} = \frac{4}{d}$

$\Rightarrow d(d-4) = 16 \quad \Rightarrow d^2 - 4d - 16 = 0$

$\Rightarrow d = \frac{1}{2}\left(4 \pm \sqrt{16 - 4(-16)}\right) = 2 \pm 2\sqrt{5} = 2\left(1 + \sqrt{5}\right)$

Therefore, in the right isosceles $\Delta\ GD_1K$, with leg $D_1G = h$ and diagonal $GK = d$, $2h^2 = d^2$

$\Rightarrow h = \dfrac{d}{\sqrt{2}} = \sqrt{2}\left(1+\sqrt{5}\right)$

So, side of octahedron is $s + h = 2\sqrt{2} + \sqrt{2}\left(1+\sqrt{5}\right) = \sqrt{2}\left(3+\sqrt{5}\right)$.

Based on the nesting of the octahedron inside the tetrahedron, the side length of tetrahedron spans twice that of the octahedron, which gives side of the tetrahedron

$$= 2(s+h) = 2\left(\sqrt{2}\left(3+\sqrt{5}\right)\right).$$

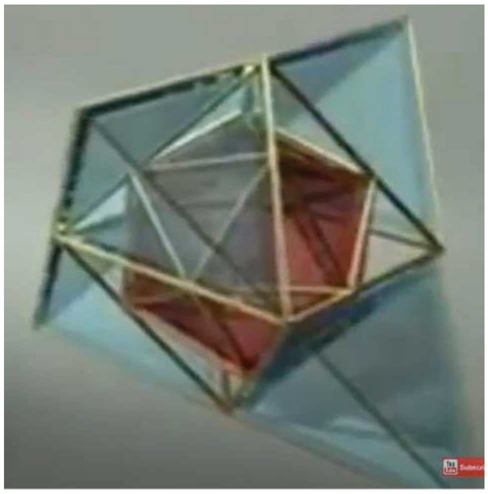

Figure 5

The side of the tetrahedron spans the diagonal of the enclosing cube. So, if side of cube is s_c, then $\sqrt{2}s_c = 2(s+h) \Rightarrow s_c = \sqrt{2}\,(s+h) = 2\left(3+\sqrt{5}\right)$.

Figure 6

The side of the cube s_c is equal to the diagonal of the pentagonal face of the dodecahedron, which means that the side of the dodecahedron s_d is given using the relation between the side and diagonal of the pentagon.

$\Rightarrow \frac{s_d}{s_c} = \frac{2}{\sqrt{5}+1}$ (from golden ratio in a pentagon)

Therefore, side of dodecahedron

$$= \frac{2}{\sqrt{5}+1} s_c = \frac{2}{\sqrt{5}+1}\sqrt{2}\,(s+h) = \frac{\sqrt{2}\left(\sqrt{5}-1\right)}{2}\,(s+h) = \frac{1}{2}\sqrt{2}\left(\sqrt{5}-1\right)\left(\sqrt{2}\left(3+\sqrt{5}\right)\right)$$
$$= 2\left(\sqrt{5}+1\right)$$

Figure 7

This entire calibration is dependent on understanding the orientation of the icosahedron inside the octahedron. Once that is understood by the reader, the remaining calculations involve simple geometry arguments.

A very interesting conclusion presents itself as one proceeds to find the dimensions of all subsequent platonic solids. It is found that these dimensions exist obeying the famous golden ratio! This stems from the fact that the golden ratio is observed in a regular pentagon, as the ratio between the side and diagonal of the pentagon.

A set of Possible Nets

1. Octahedron

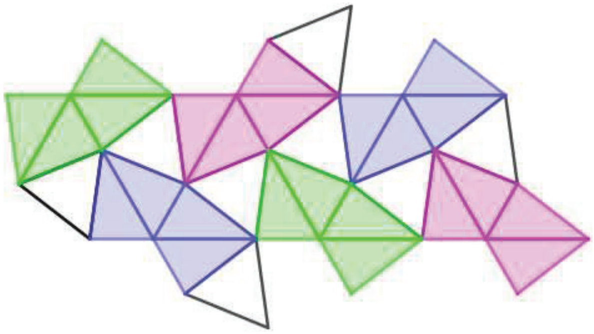

Figure 8

2. Tetrahedron

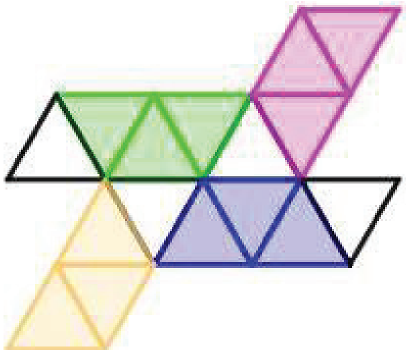

Figure 9

3. Cube

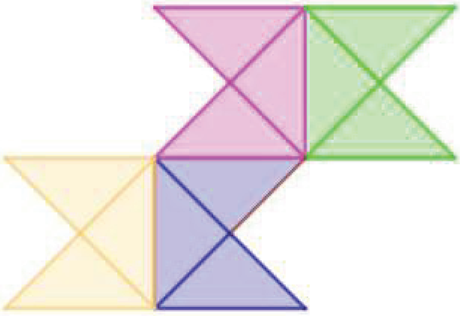

Figure 10

4. Dodecahedron

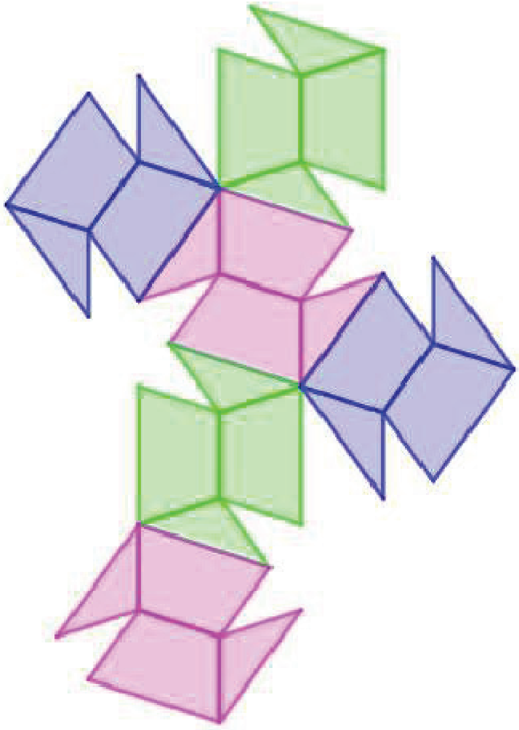

Figure 11

To create the above set of nets, I took the help of the referenced video to understand how each of the solids was seated within the other. It was an interesting task to trace possible paths to arrive at these nets while having known only how they were nested within one another. After finding a possible net for the octahedron, for example, a pattern of alternating triangles used to generate the entire solid was observable. Such a pattern could be replicated for other nets as well (except the dodecahedron where trapeziums also had to be included to form the pentagons).

References

1. https://youtu.be/gwxQfujwWrw
2. http://nonagon.org/ExLibris/sites/default/files/pdf/Kepler-Nested-Platonic-Solids.pdf

Note from Math Space

The author, Vanshika, explored the nested Platonic solids based on the video she mentioned above. The assignment involved understanding the nesting for each pair of solids and thereby finding the relative lengths of the sides of each pair. But she went beyond that! She came up with two nets, viz. those of the tetrahedron and the cube such that the coloured triangles of each net reveal the nesting of the corresponding pair of solids. This inspired us to take the work forward and create the remaining two nets, those of the octahedron and the dodecahedron. In doing the above we observed how the usual nets can be partitioned and then the resulting polygons moved around to cluster together colour-wise. Here are the complete set of nets – the usual and the new:

Octahedron net

Sides of equilateral triangles = medium sides of acute scalene triangles = 4cm

Shortest sides of scalene triangles $= 2\sqrt{2}$cm ≈ 2.83cm

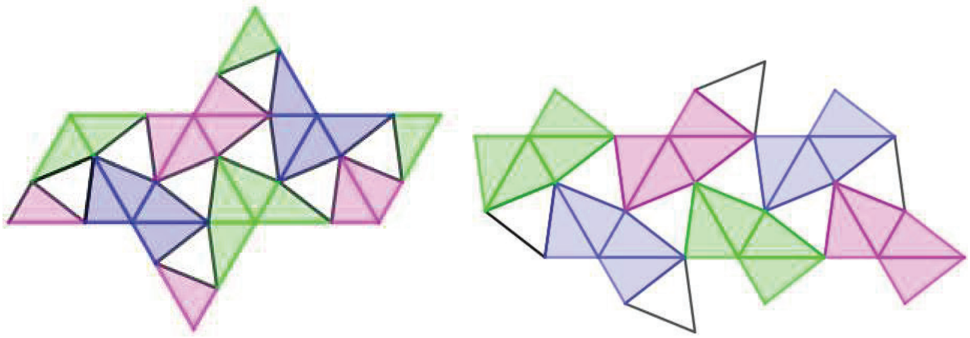

Figure 12

Tetrahedron net

Sides of smallest equilateral triangles $= \sqrt{2}\left(3 + \sqrt{5}\right)$ cm ≈ 7.40cm

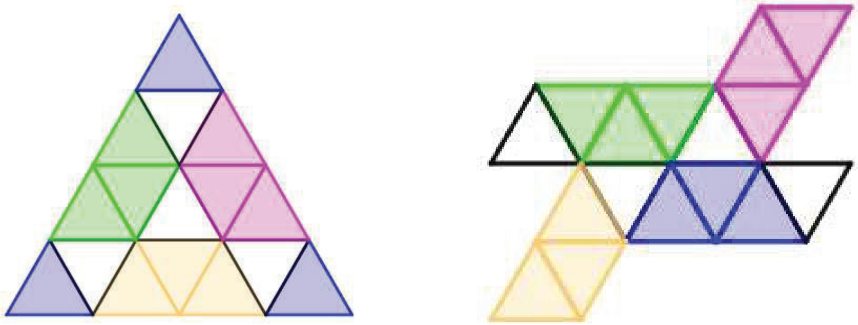

Figure 13

Cube net

Sides of squares = legs of right isosceles triangles $= 2\left(3 + \sqrt{5}\right)$ cm ≈ 10.47cm

Diagonals of squares = hypotenuse of right isosceles triangles $= 2\sqrt{2}\left(3 + \sqrt{5}\right)$ cm ≈ 14.81cm

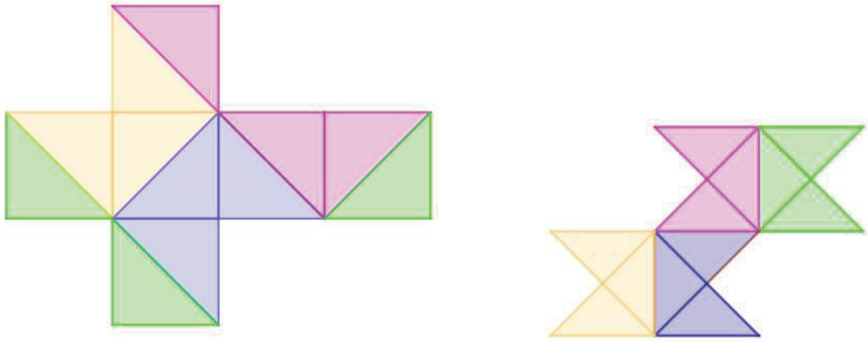

Figure 14

Dodecahedron net

Sides of regular pentagons = shorter sides of obtuse isosceles triangles = shorter sides of isosceles trapeziums $= 2\left(3+\sqrt{5}\right)$ cm $\approx$ 10.47cm

Diagonal of regular pentagons = longest side of obtuse isosceles triangles = longest side of isosceles trapeziums $= 2\left(1+\sqrt{5}\right)$ cm $\approx$ 6.47cm

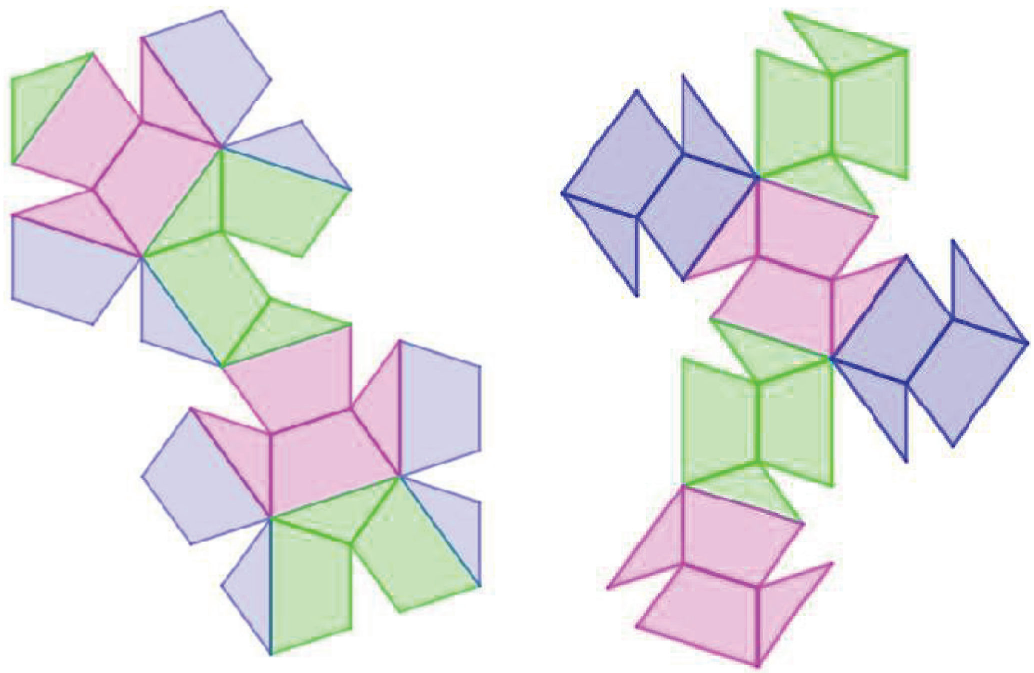

Figure 15

These nets enabled us to make physical models of the solids in question. The earlier attempt to make such models involved making the following solids separately.

(1) Icosahedron

(2) Octahedron: 6 concave hexahedrons with 2 consecutive faces as equilateral triangle, identical to the faces of the icosahedron, and the remaining 4 faces as congruent acute scalene triangles

(3) Tetrahedron: 4 regular tetrahedrons each with sides equal to that of the octahedron

(4) Cube: 4 pyramids with equilateral triangle, whose sides equal to those of the tetrahedron, as base and right isosceles triangles as the remaining faces

(5) Dodecahedron: 6 pentahedrons with square bases, identical to the faces of the cube, the remaining being 2 pairs of congruent polygons – isosceles trapeziums and obtuse isosceles triangles, alternating with each other – all remaining sides of these 4 polygons being equal to the sides of the dodecahedron.

Now, these pentahedrons can be taped along some edges to form a shell that encloses the cube and generates the dodecahedron. Similarly, the pyramids can be joined to form a shell enclosing the tetrahedron and making the cube. The 4 smaller tetrahedrons can be placed around and above the octahedron to generate the tetrahedron inside the cube. But it is quite difficult to connect the hexahedrons to form the shell for the octahedron. This is mainly because these concave solids connect only at some of their vertices. This problem is resolved thanks to the nets improvised and inspired by the author. Note that the same acute scalene triangles are in the nets in Figure 12 as well. The rest of the polygons (isosceles and equilateral triangles and isosceles trapeziums) are all symmetric and therefore easier to construct. But the construction of this acute scalene triangles revealed some unexpected surprises! As illustrated in Figure 3, the shortest side, e.g., AP of this triangle is $2\sqrt{2}$cm while the medium side is 4cm. Also, the angle opposite to the medium side is 60°. Now, $2\sqrt{2}$cm can be easily obtained from 4cm using a right isosceles triangle. So, by Side-Side-Angle (SSA) construction, one can easily generate this acute scalene triangle. On the other hand, the longest side is $\sqrt{2}\left(1+\sqrt{5}\right)$ which is more complicated to construct. So, SSA is a preferred way than SSS. Note that SSA construction (where (i) an angle, (ii) an adjacent side and (iii) the opposite side are given) is no longer part of many textbooks. This is possibly because it fails to be a congruency criterion like the rest, i.e., SAS, SSS, ASA (and AAS) and RHS. If the given angle is acute and the side opposite to it is shorter than the given adjacent side, then there can be 2 triangles – one acute, e.g., $\triangle ABC$ and one obtuse, e.g., $\triangle ABD$ both satisfying the SSA criterion $\measuredangle A, AB, BC = BD$ (see Figure 16). In this case however, since the adjacent side $2\sqrt{2}$cm < 4cm, the opposite side, the construction generates a unique triangle. One can easily find applications of SAS, SSS, ASA (AAS) and RHS in constructing various quadrilaterals, in particular. We were pleasantly surprised to find an application of SSA while exploring nested Platonic solids!

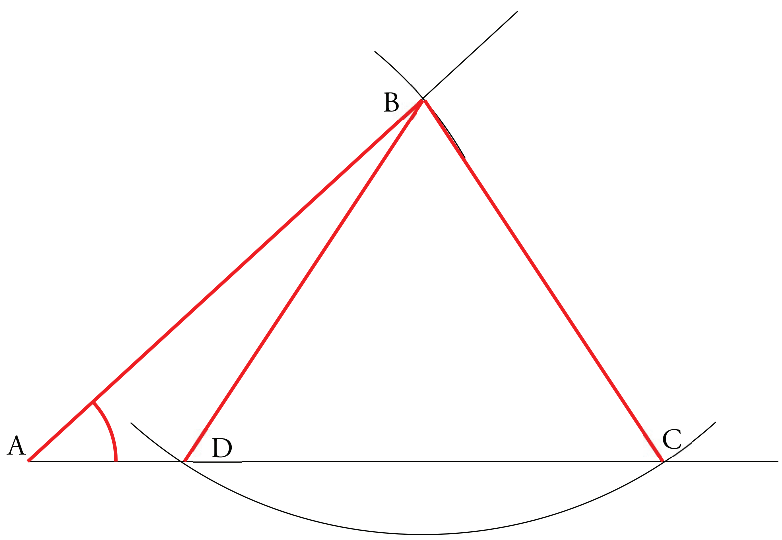

Figure 16

VANSHIKA MITTAL is currently a third year undergraduate student pursuing B.Sc (Honours) Mathematics at Azim Premji University, Bangalore, though she worked on this project at the end of her first year. She is currently working on a project to understand the concept of Fractional Calculus as part of the Honours program at the university. She finds topics in Analysis particularly interesting to study. Apart from mathematics, she also likes playing the violin and understanding patterns between Mathematics and music. Vanshika may be contacted at vanshikamittal64@gmail.com