ClassRoom

# Sums of Powers of Any Composite Number

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In 1997, David B. Sher visually established a well-known result on sums of powers of four [1, p.135]. He proved for any $r \in \mathbb{N}$,

$$4^r - 1 = 3\sum_{k=0}^{r-1} 4^k$$

i.e.,

$$\frac{4^r - 1}{4 - 1} = \sum_{k=0}^{r-1} 4^k$$

In 2021, extending the same technique in three dimensions another results for sums of powers of eight [2] is established.

$$8^r - 1 = 7\sum_{k=0}^{r-1} 8^k$$

i.e.,

$$\frac{8^r - 1}{8 - 1} = \sum_{k=0}^{r-1} 8^k$$

The primary aim of this short note is to visually establish sums of powers of any composite number.

For $9 = 3 \times 3$, we illustrate the above result in the following diagram. The above diagram also generalizes the result David B. Sher.

*Keywords: Series, Composite numbers, patterns, visualisation.*

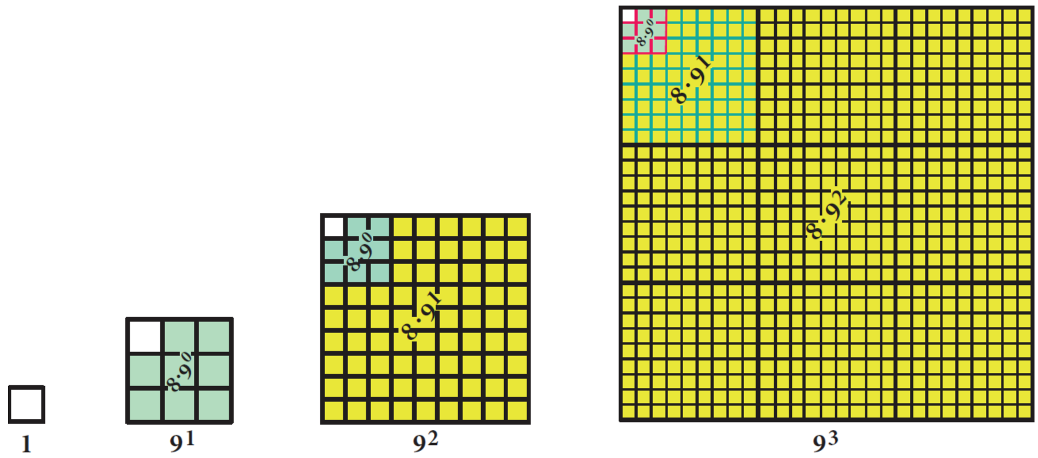


Figure 1

Note that, in the $9^3$ grid, $8 \times 9^2$ is represented by 8 squares with $9^2 = 81$ smaller squares.

$$9^r = 1 + 8.9^0 + 8.9^1 + 8.9^2 + \ldots + 8.9^{r-1}$$

$$\Rightarrow 9^r - 1 = 8\left[9^0 + 9^1 + 9^2 + \ldots + 9^{r-1}\right]$$

$$\Rightarrow \frac{9^r - 1}{8} = \sum_{i=0}^{r-1} 9^i,$$

$$\sum_{i=0}^{r-1} 9^i = \frac{9^r - 1}{9 - 1}$$

**Note:** Let $x = m \times n$, be any composite number, $m$, $n$ be two natural numbers, then

$$\sum_{i=0}^{r-1} x^i = \frac{x^r - 1}{x - 1}$$

Now, I extend the above result in 3D for the composite number 27.

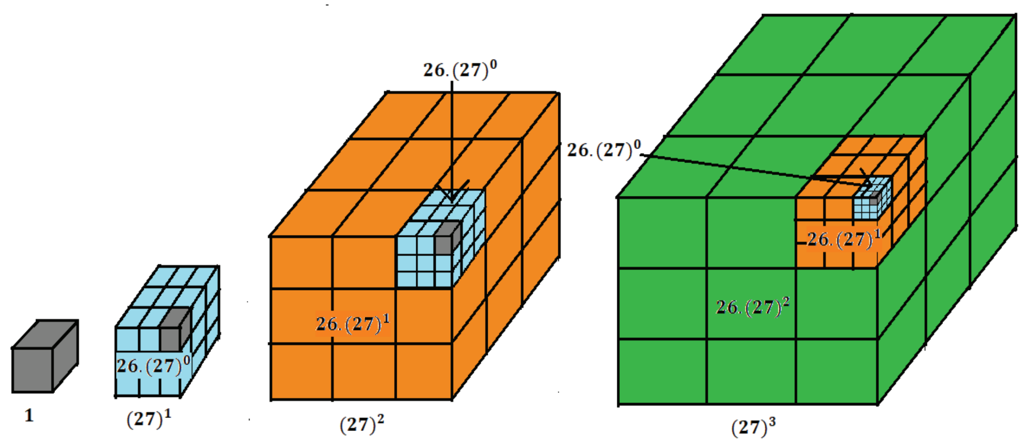


Figure 2

For $27 = 3 \times 3 \times 3$, we illustrate the above result in Figure 2.

Please note that, in the $27^3$ grid, $26 \times 27^2$ is represented by 26 cubes with $27^2 = 729$ smaller cubes.

$$(27)^r = 1 + 26.\,(27)^0 + 26.\,(27)^1 + 26.\,(27)^2 + \ldots + 26.\,(27)^{r-1}$$
$$\Rightarrow (27)^r - 1 = 26\left[(27)^0 + (27)^1 + (27)^2 + \ldots + (27)^{r-1}\right]$$
$$\Rightarrow \frac{(27)^r - 1}{26} = \sum_{i=0}^{r-1} (27)^i$$

**Note:** Let $x = m \times n \times p$, be any composite number, $m, n, p$ be any three natural numbers. Then,

$$\sum_{i=0}^{r-1} x^i = \frac{x^r - 1}{x - 1}$$

For any composite number $x = m_1.m_2 \ldots m_k$, one can use similar techniques to generalize the above two results in $k$-dimensions $(k \geq 2)$.

In this short note, I have tried to generalize [1] and [2] for any composite $x$ with $x = m_1.m_2 \ldots m_k$.

$$\sum_{i=0}^{r-1} x^i = \frac{x^r - 1}{x - 1}$$

## Reference


1. Roger B. Nelsen. Proofs without Words III: More Exercises in Visual Thinking. MAA Press, 2015.
2. Rajib Mukherjee, Proofs without Words: Sums of Powers of Eight Mathematical Intelligencer **43**, 60-61 (2021) (https://doi.org/10.1007/s00283-021-10087-5).



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