

Amazing Shapes using Factorial Digits

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The factorial of a natural number n is the product of the positive integers less than or equal to n . It is written as $n!$ and pronounced 'n factorial'. The first few factorials for $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10...$ are 1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800... etc. $n!$ gives the number of ways in which n objects can be permuted. The special case $0!$ is defined to have value $0! = 1$.

The number of digits in factorials grows very fast. For example, $6!$ (i.e., 720) consists of 3 digits, but the number of digits grows to 23 for $23!$ i.e., 25852016738884976640000. Interestingly, the digits of factorials can be represented in many amazing shapes such as triangle, rhombus, hexagon, etc., but for this, it is necessary that the number of digits in $n!$ must be such that it can represent that shape. In this paper, you can find as to how the factorials with required number of digits for the desired shape can be obtained.

For geometrical shapes like triangle, rhombus, hexagon, octagon, two sides are considered equal if the number of digits placed on each side is equal. So, for equilateral triangle, the number of digits of each of the three sides must be equal. The number of digits in a factorial which are required to decide/draw any shape can be computed as follows:

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The number of digits in the base 10 representation of a number x is given by

$\lfloor \log_{10} x \rfloor + 1$, where $\lfloor m \rfloor$ is the floor of m , the largest integer less than or equal to m . The log of the factorial function is easier to compute than the factorial itself. For any $n > 0$, the number of digits in $n!$ i.e. $d(n!) = \lfloor \log_{10} n! \rfloor + 1$.

For example, $d(23!) = \lfloor \log_{10} 23! \rfloor + 1 = \lfloor 22.41 \rfloor + 1 = 22 + 1 = 23$. Table 1 gives several examples.

S.No.	n	Number of digits in $n!$	S.No.	n	Number of digits in $n!$
1	5	3	20	335	703
2	6	3	21	350	741
3	9	6	22	381	820
4	13	10	23	413	903
5	17	15	24	446	990
6	32	36	25	463	1035
7	38	45	26	480	1081
8	44	55	27	570	1326
9	65	91	28	589	1378
10	106	171	29	608	1431
11	125	210	30	647	1540
12	135	231	31	667	1596
13	156	276	32	687	1653
14	178	325	33	728	1770
15	201	378	34	749	1830
16	213	406	35	770	1891
17	278	561	36	880	2211
18	292	595	37	996	2556
19	306	630			

Table 1

Equilateral triangles from factorial digits

It can be seen from Figure 1 that the first row consists of 1 digit, second row of 2 digits, third row of 3 digits and so on. So, the n^{th} row consists of n digits. So, the number of digits in any triangle is the partial sum of the series $1 + 2 + 3 + 4 + 5 + \dots + n$, which is always a triangular number given by $\frac{n(n+1)}{2}$. So, if the number of digits in $n!$ is a triangular number, then the digits of that factorial can be represented in the form of triangles as shown in Figure 1. There are 37 factorials below 1000! for which the number of digits is a triangular number greater than 1 and these are shown in Table 1. It can be seen that this triangular shape is actually an equilateral triangle that has all three sides of equal length (i.e., equal number of digits).

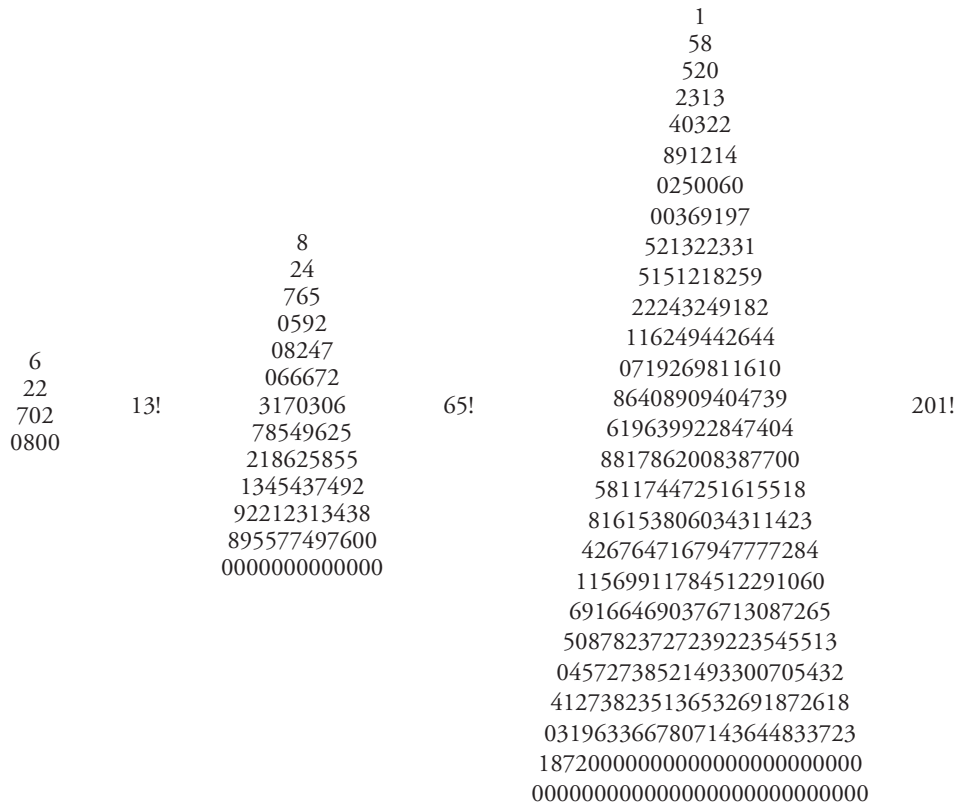


Figure 1

Rhombus from factorial digits

It can be seen from Figure 2 that rhombus can be represented as a combination of two triangles, one with $\frac{n(n+1)}{2}$ digits placed upside down below the base of the other triangle with $\frac{(n+1)(n+2)}{2}$ digits. Since the sum of two consecutive triangular numbers is always a perfect square, if the number of digits in a factorial is a perfect square, then the digits of that factorial can be represented in the form of rhombus as shown in Figure 2. It can be seen that this rhombus shape has all four sides of equal length (i.e., equal number of digits) and two unequal diagonals. There are 20 factorials below 1000! for which the number of digits is a square number greater than 1 and these are shown in Table 2.

S.No.	n	Number of digits in $n!$	S.No.	n	Number of digits in $n!$
1	7	4	11	284	576
2	12	9	12	304	625
3	18	16	13	367	784
4	32	36	14	389	841
5	59	81	15	435	961
6	81	121	16	483	1089
7	105	169	17	508	1156
8	132	225	18	697	1681
9	228	441	19	726	1764
10	265	529	20	944	2401

Table 2

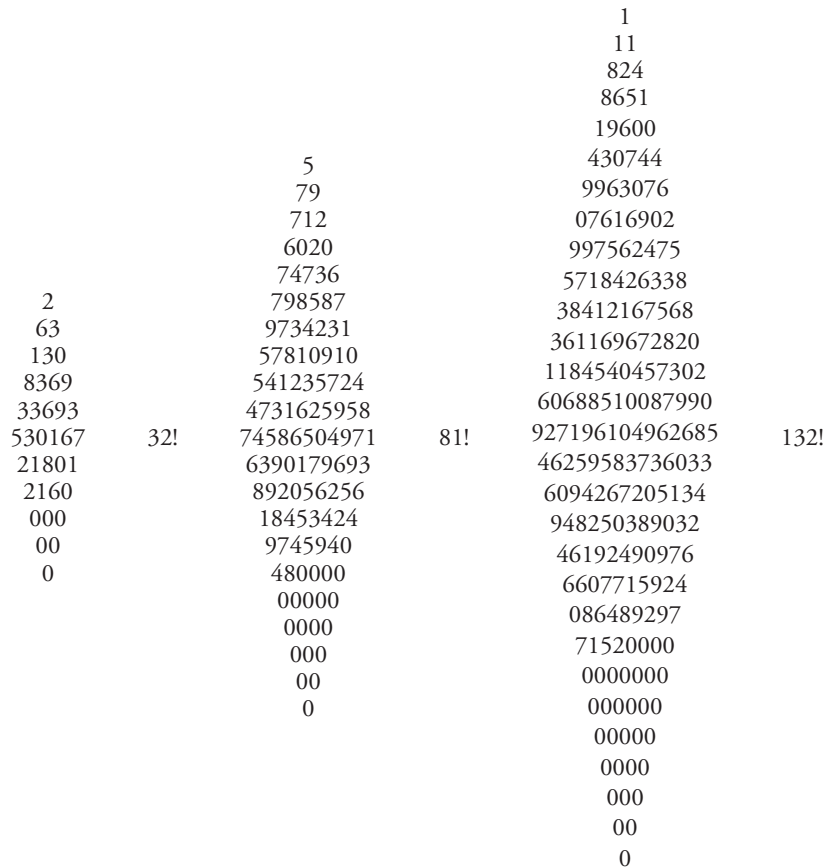


Figure 2

Hexagon from factorial digits

It can be seen from Figure 3 that the first row consists of d digits (where d is the number of digits in each side), each subsequent row consists of 2 digits more than the previous row till we reach the d^{th} row. After the d^{th} row, each subsequent row consists of 2 digits less than the previous row till we reach the last row that is the $(2d - 1)^{\text{th}}$ row. So, the number of digits in any such hexagonal shape is $4 \times d \times d - 5 \times d + 2$. So, if the number of digits in n factorial is equal to $4 \times d \times d - 5 \times d + 2$, then the digits of that factorial can be represented in the form of a hexagon as shown in Figure 3. As shown in Table 3, there are 18 factorials below 2000! which have $4 \times d \times d - 5 \times d + 2$ digits. It shall be noted that each side of the hexagon consists of d digits.

S.No.	n	Number of digits in $n!$	S.No.	n	Number of digits in $n!$
1	11	8	10	477	1073
2	23	23	11	527	1208
3	57	77	12	690	1661
4	78	116	13	936	2377
5	129	218	14	1142	2998
6	158	281	15	1289	3452
7	190	352	16	1444	3938
8	224	431	17	1691	4727
9	299	613	18	1955	5588

Table 3

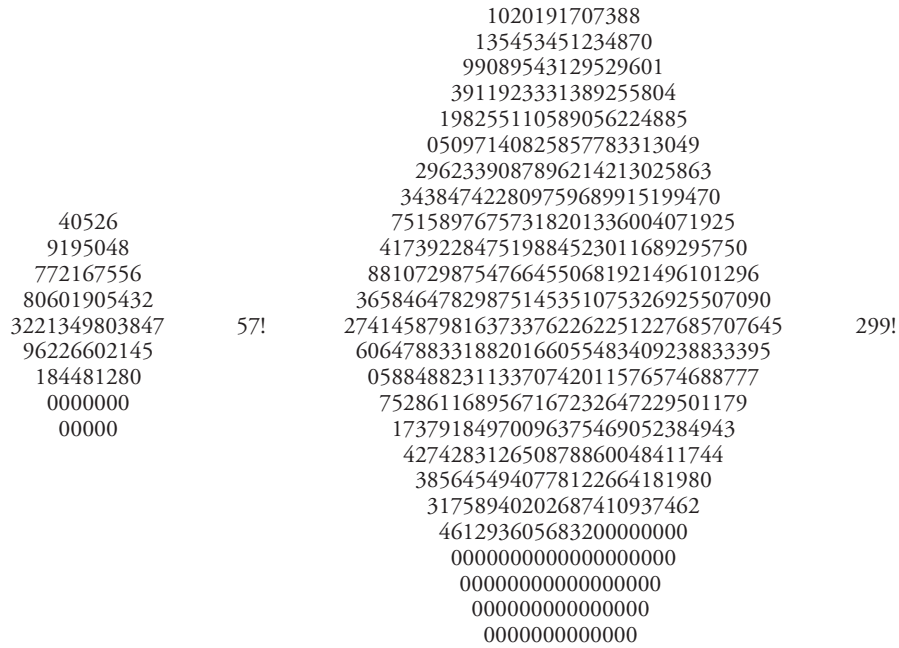


Figure 3

Octagon from factorial digits

If d is the number of digits in each side and number of digits in a factorial is equal to $7 \times d \times d - 10 \times d + 4$, then digits of that factorial can be represented in the form of an octagon as shown in figure 4. As shown in Table 4, there are 19 factorials below 5000! which have $7 \times d \times d - 10 \times d + 4$ digits. It shall be noted that each side of the octagon consists of d digits.

S.No.	n	Number of digits in $n!$
1	33	37
2	85	129
3	245	481
4	350	741
5	471	1057
6	681	1636
7	924	2341
8	1012	2604
9	1297	3477
10	1399	3796
11	1613	4476
12	1725	4837
13	1959	5601
14	2081	6004
15	2206	6421
16	2601	7756
17	2739	8229
18	4464	14356
19	4639	14997

Table 4

