Amazing Shapes using Factorial Digits

SHYAM SUNDER GUPTA

The factorial of a natural number n is the product of the positive integers less than or equal to n. It is written as n! and pronounced 'n factorial'. The first few factorials for n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10... are 1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800... etc. n! gives the number of ways in which n objects can be permuted. The special case 0! is defined to have value 0! = 1.

The number of digits in factorials grows very fast. For example, 6! (i.e., 720) consists of 3 digits, but the number of digits grows to 23 for 23! i.e., 25852016738884976640000. Interestingly, the digits of factorials can be represented in many amazing shapes such as triangle, rhombus, hexagon, etc., but for this, it is necessary that the number of digits in *n*! must be such that it **can** represent that shape. In this paper, you can find as to how the factorials with required number of digits for the desired shape can be obtained.

For geometrical shapes like triangle, rhombus, hexagon, octagon, two sides are considered equal if the number of digits placed on each side is equal. So, for equilateral triangle, the number of digits of each of the three sides must be equal. The number of digits in a factorial which are required to decide/draw any shape can be computed as follows:

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The number of digits in the base 10 representation of a number x is given by

$\lfloor \log_{10} x \rfloor + 1$, where $\lfloor m \rfloor$ is the floor of m, the largest integer less than or equal to m. The log of the factorial function is easier to compute than the factorial itself. For any $n > 0$, the number of digits in $n!$ i.e. $d(n!) = \lfloor \log_{10} n! \rfloor + 1$.

For example, $d(23!) = \lfloor \log_{10} n! \rfloor + 1 = \lfloor 22.41 \rfloor + 1 = 22 + 1 = 23$. Table 1 gives several examples.

S.No.	n	Number of digits in $n!$	S.No.	n	Number of digits in $n!$
1	5	3	20	335	703
2	6	3	21	350	741
3	9	6	22	381	820
4	13	10	23	413	903
5	17	15	24	446	990
6	32	36	25	463	1035
7	38	45	26	480	1081
8	44	55	27	570	1326
9	65	91	28	589	1378
10	106	171	29	608	1431
11	125	210	30	647	1540
12	135	231	31	667	1596
13	156	276	32	687	1653
14	178	325	33	728	1770
15	201	378	34	749	1830
16	213	406	35	770	1891
17	278	561	36	880	2211
18	292	595	37	996	2556
19	306	630			

Table 1

Equilateral triangles from factorial digits

It can be seen from Figure 1 that the first row consists of 1 digit, second row of 2 digits, third row of 3 digits and so on. So, the n^{th} row consists of n digits. So, the number of digits in any triangle is the partial sum of the series $1 + 2 + 3 + 4 + 5 + \ldots n$, which is always a triangular number given by $\frac{n(n+1)}{2}$. So, if the number of digits in $n!$ is a triangular number, then the digits of that factorial can be represented in the form of triangles as shown in Figure 1. There are 37 factorials below 1000! for which the number of digits is a triangular number greater than 1 and these are shown in Table 1. It can be seen that this triangular shape is actually an equilateral triangle that has all three sides of equal length (i.e., equal number of digits).

6
22
702
0800

13!

8
24
765
0592
08247
066672
3170306
78549625
218625855
1345437492
92212313438
895577497600
0000000000000

65!

1
58
520
2313
40322
891214
0250060
00369197
521322331
5151218259
22243249182
116249442644
0719269811610
86408909404739
619639922847404
8817862008387700
58117447251615518
816153806034311423
4267647167947777284
11569911784512291060
691664690376713087265
5087823727239223545513
04572738521493300705432
412738235136532691872618
0319633667807143644833723
18720000000000000000000000
000000000000000000000000000

201!

Figure 1

Rhombus from factorial digits

It can be seen from Figure 2 that rhombus can be represented as a combination of two triangles, one with $\frac{n(n+1)}{2}$ digits placed upside down below the base of the other triangle with $\frac{(n+1)(n+2)}{2}$ digits. Since the sum of two consecutive triangular numbers is always a perfect square, if the number of digits in a factorial is a perfect square, then the digits of that factorial can be represented in the form of rhombus as shown in Figure 2. It can be seen that this rhombus shape has all four sides of equal length (i.e., equal number of digits) and two unequal diagonals. There are 20 factorials below 1000! for which the number of digits is a square number greater than 1 and these are shown in Table 2.

S.No.	n	Number of digits in $n!$	S.No.	n	Number of digits in $n!$
1	7	4	11	284	576
2	12	9	12	304	625
3	18	16	13	367	784
4	32	36	14	389	841
5	59	81	15	435	961
6	81	121	16	483	1089
7	105	169	17	508	1156
8	132	225	18	697	1681
9	228	441	19	726	1764
10	265	529	20	944	2401

Table 2

2
63
130
8369
33693
530167
21801
2160
000
00
0

32!

5
79
712
6020
74736
798587
9734231
57810910
541235724
4731625958
74586504971
6390179693
892056256
18453424
9745940
480000
00000
0000
000
00
0

81!

1
11
824
8651
19600
430744
9963076
07616902
997562475
5718426338
38412167568
361169672820
1184540457302
60688510087990
927196104962685
46259583736033
6094267205134
948250389032
46192490976
6607715924
086489297
71520000
0000000
000000
00000
0000
000
00
0

132!

Figure 2

Hexagon from factorial digits

It can be seen from Figure 3 that the first row consists of d digits (where d is the number of digits in each side), each subsequent row consists of 2 digits more than the previous row till we reach the d^{th} row. After the d^{th} row, each subsequent row consists of 2 digits less than the previous row till we reach the last row that is the $(2d-1)^{\text{th}}$ row. So, the number of digits in any such hexagonal shape is $4 \times d \times d - 5 \times d + 2$. So, if the number of digits in n factorial is equal to $4 \times d \times d - 5 \times d + 2$, then the digits of that factorial can be represented in the form of a hexagon as shown in Figure 3. As shown in Table 3, there are 18 factorials below 2000! which have $4 \times d \times d - 5 \times d + 2$ digits. It shall be noted that each side of the hexagon consists of d digits.

S.No.	n	Number of digits in n!	S.No.	n	Number of digits in n!
1	11	8	10	477	1073
2	23	23	11	527	1208
3	57	77	12	690	1661
4	78	116	13	936	2377
5	129	218	14	1142	2998
6	158	281	15	1289	3452
7	190	352	16	1444	3938
8	224	431	17	1691	4727
9	299	613	18	1955	5588

Table 3

40526
9195048
772167556
80601905432
3221349803847
96226602145
184481280
0000000
00000

57!

1020191707388
135453451234870
99089543129529601
3911923331389255804
198255110589056224885
05097140825857783313049
2962339087896214213025863
343847422809759689915199470
75158976757318201336004071925
4173922847519884523011689295750
881072987547664550681921496101296
36584647829875145351075326925507090
2741458798163733762262251227685707645
60647883318820166055483409238833395
058848823113370742011576574688777
7528611689567167232647229501179
17379184970096375469052384943
427428312650878860048411744
3856454940778122664181980
31758940202687410937462
4612936056832000000000
0000000000000000000
00000000000000000
000000000000000
0000000000000

299!

Figure 3

Octagon from factorial digits

If d is the number of digits in each side and number of digits in a factorial is equal to $7 \times d \times d - 10 \times d + 4$, then digits of that factorial can be represented in the form of an octagon as shown in figure 4. As shown in Table 4, there are 19 factorials below 5000! which have $7 \times d \times d - 10 \times d + 4$ digits. It shall be noted that each side of the octagon consists of d digits.

S.No.	n	Number of digits in n!
1	33	37
2	85	129
3	245	481
4	350	741
5	471	1057
6	681	1636
7	924	2341
8	1012	2604
9	1297	3477
10	1399	3796
11	1613	4476
12	1725	4837
13	1959	5601
14	2081	6004
15	2206	6421
16	2601	7756
17	2739	8229
18	4464	14356
19	4639	14997

Table 4

868
33176
1881188
6495518
1944012
80000
000

33!

28171
0411438
055027694
94794422606
1159480056634
3305742064051
0191275256002
6159795933451
0402864523409
24018275123
200000000
0000000
00000

85!

344638108
85481846673
2677079087595
192200697695137
55823846110036119
8163952978235186876
380414323767237937984
68686285775275996090439
4401084934335518382691657
9274876093562728501050027
8210756098320353036549967
4936175319516301276907070
0485723141551669720900953
7280115585840030353759282
1265584694028136706112564
5619508966404523085638743
7632569808704944651568598
19224514274661087972929
242817912092409671474
4916317976775473389
10924800000000000
000000000000000
0000000000000
00000000000
000000000

245!

Figure 4

Isosceles Triangle from factorial digits

For an isosceles triangle, it may be seen that the first row consists of 1 digit, 2nd row consists of 3 digits, 3rd row consists of 5 digits and so on. So, n^{th} row consists of $(2n-1)$ digits. So, the number of digits is the partial sum of series 1 + 3 + 5 + 7 + 9 +…. which is always a perfect square. Can you use the factorials in Table 2 to draw isosceles triangles?

Using factorial digits, it is possible to draw various other shapes such as cross (+), E, F, I, L, T etc. It will be an interesting pastime to attempt to draw the above mentioned shapes and find out the factorials with the required number of digits for the desired shapes.

SHYAM SUNDAR GUPTA served in the Indian Railways for 35 years, and retired as Principal Chief Engineer in 2018. Popularising mathematics through number recreations has been his passion for more than forty years. His contributions have been published in National and International journals/books. He is the author of the book "Creative Puzzles to Ignite Your Mind" published by Springer in March 2023 and co-author of the book 'Civil Engineering through Objective Type Questions' published in 1985. E-mail: guptass@rediffmail.com Web page: http://www.shyamsundergupta.com