ClassRoom

Connections between Paper Folding, Geometry and Proof

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Let us consider a rectangle ABCD. We create another rectangle ABFE within it by joining the midpoints F and E of the breadths of the original rectangle ABCD. Then the diagonal AC of the original rectangle and the diagonal BE of the second rectangle (ABFE) intersect at a point O in such a way that if we draw a straight line through O parallel to DC, which intersects the breadths AD & BC of the original rectangle at G and H respectively, then we get a third rectangle ABHG whose breadth will be **one-third (1/3)** the breadth of the original rectangle."

In general, if we apply this same process to the newly obtained rectangle, then we will get a new rectangle whose breadth will be again one third of the previous rectangle i.e., this will be a continuous process. Further we will get the same result if we get the intersection point by involving other diagonals of rectangles i.e., BD and AF respectively.

Note: This intriguing statement was made by my mentor. It happened as described below.

Keywords: Paper Folding, Geometry, Exploration, Verification, Reasoning, Proof.

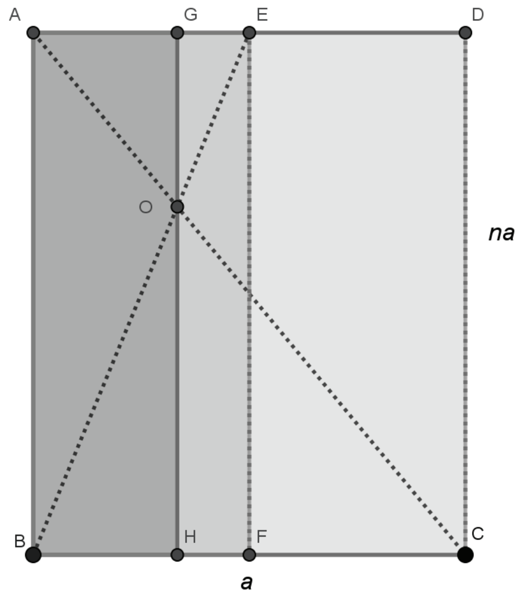

Figure 1.

One day, during my associate program, after completing my school assignments, I arrived at the District Institute. It was on that day that my mentor asked us to verify this statement through folding an A-4 size paper. Intrigued, I replicated the paper folding process with an A-4 size paper and verified its dimensions using a ruler. To my surprise, the results aligned perfectly.

This ignited a discussion between my mentor and myself, prompting him to challenge me: "Can you prove it?" Eager to showcase my mathematical prowess and passion, I eagerly accepted the challenge. I meticulously depicted the paper-folding figure in my notebook and engaged in intense contemplation on how to formulate a proof. When I proved it, I presented it to my mentor.

Buoyed by this success, I set out to articulate the theorem and its corresponding proof in a succinct manner. This endeavour deepened my comprehension of the intricate connection between paper folding, geometry, and mathematical analysis. It became evident that geometry serves as a foundation upon which mathematical analysis can flourish, with geometric concepts often serving as a springboard for the development of intricate mathematical analyses. Here is my proof.

Proof : Let us consider the length of the rectangle (i.e., the longer side) $AB = DC = na$ unit, and breadth $BC = AD = a$ unit, where n is any real number.

Applying Pythagoras theorem in triangle ABE , we get

$$BE^2 = AB^2 + AE^2 = (na)^2 + \left(\frac{a}{2}\right)^2 = (4n^2+1)\frac{a^2}{4};$$

So

$$BE = \sqrt{(4n^2+1)}\frac{a}{2} \tag{1}$$

Since ΔBOH & ΔBEF are similar, we have –

$$\frac{BH}{BF} = \frac{BO}{BE} = \frac{HO}{FE} \tag{2}$$

Using $BF = \frac{BC}{2} = \frac{\mathrm{a}}{2}$, in $\frac{BH}{BF} = \frac{BO}{BE}$, we have $\frac{2BH}{a} = \frac{BO}{BE}$

$$\Rightarrow BO = \sqrt{(4n^2+1)}BH \qquad \{\text{using equation (1)}\} \tag{3}$$

Now using equation (2) again we have –

$$\frac{BO}{BE} = \frac{HO}{FE}, \frac{2BO}{\sqrt{(4n^2+1)}} = \frac{HO}{n} \qquad \{\text{since } FE = AB = na \text{ \& using equation (1)}\}$$

which gives us,

$$HO = 2n\,BH \tag{4}$$

Similarly, ΔAOE & ΔCOB are similar, we have –

$$\frac{OE}{OB} = \frac{AE}{CB}, OB = 2OE \left(\text{since } AE = \frac{CB}{2}\right)$$

So, from Figure 1,

$$BE = 3OE = \frac{\mathrm{a}}{2}\sqrt{(4n^2+1)} \quad OE = \frac{\mathrm{a}}{6}\sqrt{(4n^2+1)} \quad OB = \frac{\mathrm{a}}{3}\sqrt{(4n^2+1)} \tag{5}$$

Now using equation (4) & (5) in ΔBOH we have –

$$BO^2 = BH^2 + OH^2 = (4n^2+1)BH^2 \quad (4n^2+1)\frac{\mathrm{a}^2}{9} = (4n^2+1)BH^2$$

Which gives us

$$BH = \frac{\mathrm{a}}{3} \qquad \text{(Hence Proved).}$$

AKASH MAURYA has a Master's degree in Mathematics from University of Allahabad, Uttar Pradesh. He has been working as an Associate with Azim Premji Foundation since August 2022. His interests includes reading and teaching mathematics, reading editorials, listening to music, etc. Akash can be contacted on the phone at +91 9598288905 and on email at akash.maurya@azimpremjifoundation.org