Two New Proofs of the Pythagorean Theorem - Part II

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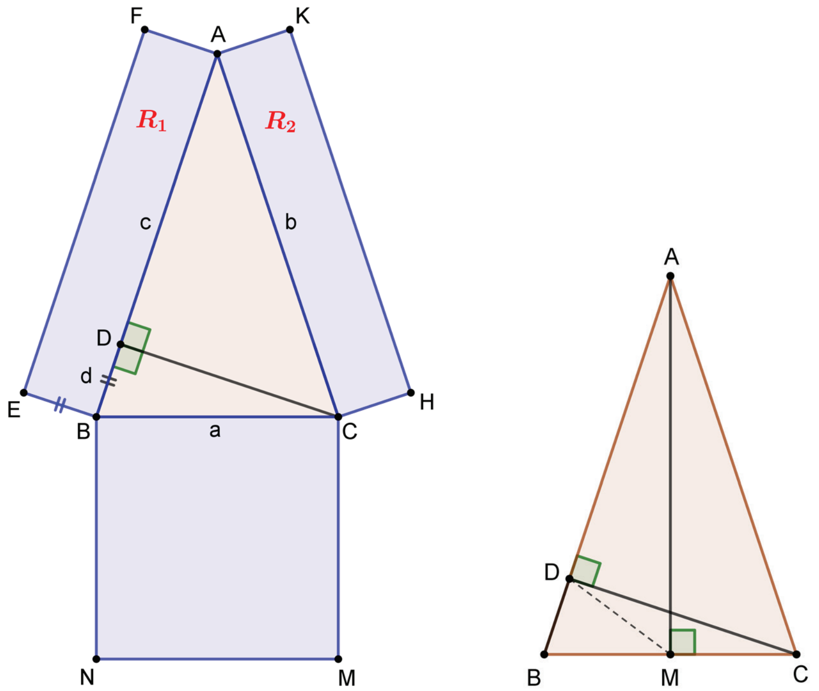
In Part I of this article which appeared in the July 2023 issue of *At Right Angles*, we stated that “[the] Pythagorean theorem (‘PT’ for short) is easily the best known result in all of mathematics. What is less well-known is the fact that among all theorems in mathematics, it holds the ‘world record’ for the number of different proofs. There is no other theorem that even comes close! (See [2] and [3].) In the book *The Pythagorean Proposition* [1] (published in 1940), the author Elisha S. Loomis lists as many as 370 different proofs of the theorem. Since that time …more proofs have appeared.” Later in the article we presented a new and very novel proof of the PT by two high-school teenagers, Calcea Johnson and Ne’Kiya Jackson, both from New Orleans, USA (see [6], [7], [8] and [9]).

Now in Part II we present an adaptation of a proof [4] by Professor Kaushik Basu, a well-known World Bank economist; he describes the proof as “new and very long” but gives a poetic and eloquent justification for adding this proof to the long list of existing proofs.

Keywords: Pythagorean theorem, Kaushik Basu, Calcea Johnson, Ne'Kiya Jackson, St. Mary's Academy, New Orleans, trigonometric proof



(a) (b)

Figure 1. The `Isosceles Lemma' and its proof

An adaptation of Kaushik Basu's "New and Very Long Proof"

As noted above, Basu describes his own proof as "very long." He starts by establishing two subsidiary results or lemmas which are of interest in themselves. While writing this article, I found that Lemma 1 ('The Isosceles Lemma') is not really needed, and Lemma 2 ('The Right-Angled Lemma') leads directly to a proof of the PT. However, for the sake of completeness, we describe both the lemmas.

Lemma 1 (The Isosceles Lemma)**.** *Let ABC be an acute-angled isosceles triangle with $AB = AC$. Draw a perpendicular CD from vertex C to side AB. Let the length of BD be d. Draw a rectangle R_1 with side AB as base and height d. Similarly, draw a rectangle R_2 with side AC as base and height d. (Of course, R_1 and R_2 are congruent to each other.) Now draw a square on side BC. Then the sum of the areas of R_1 and R_2 is equal to the area of the square.*

Proof. See Figure 1 (a). Using the language of algebra rather than geometry, we need to prove that $a^2 = cd + bd$, i.e., $a^2 = 2cd$, since $b = c$.

Drop a perpendicular AM from vertex A to the base BC; see Figure 1 (b). Since the triangle is isosceles, M lies at the midpoint of BC. Now observe that quadrilateral $ADMC$ is cyclic. This is true because $\measuredangle ADC = \measuredangle AMC$ (both are right angles). Hence we have (by the intersecting chords theorem):

$$BD \cdot BA = BM \cdot BC, \tag{1}$$

i.e., $d \cdot c = \frac{1}{2} a \cdot a$. It follows that $a^2 = 2cd$, as was to be proved. $\square$

Remark. The Isosceles Lemma is of interest in itself, independent of its role in the proof of the PT. As we shall see, we do not really need this result to prove the PT.

Lemma 2 (The Right-Angled Lemma)**.** *Let a right-angled triangle ABC be given, with $\measuredangle B = 90°$. Locate a point D on the hypotenuse AC such that $AD = AB$. Let $CD = d$. Draw a rectangle R_3 with AC as base and height d. Draw a rectangle R_4 with AB as base and height d. Now draw a square on side BC. Then the sum of the areas of R_3 and R_4 is equal to the area of the square.*

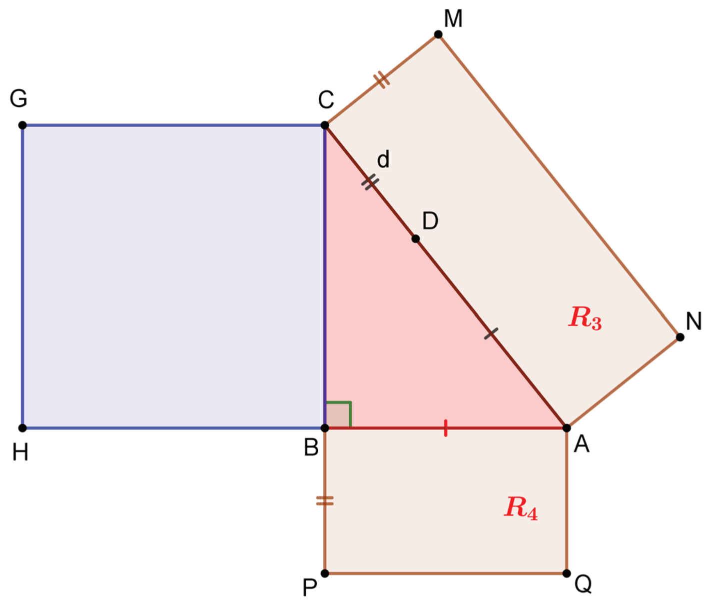


Figure 2. The Right-Angled Lemma. By construction, $AD = AB$; $CM = CD$; $BP = CD$.

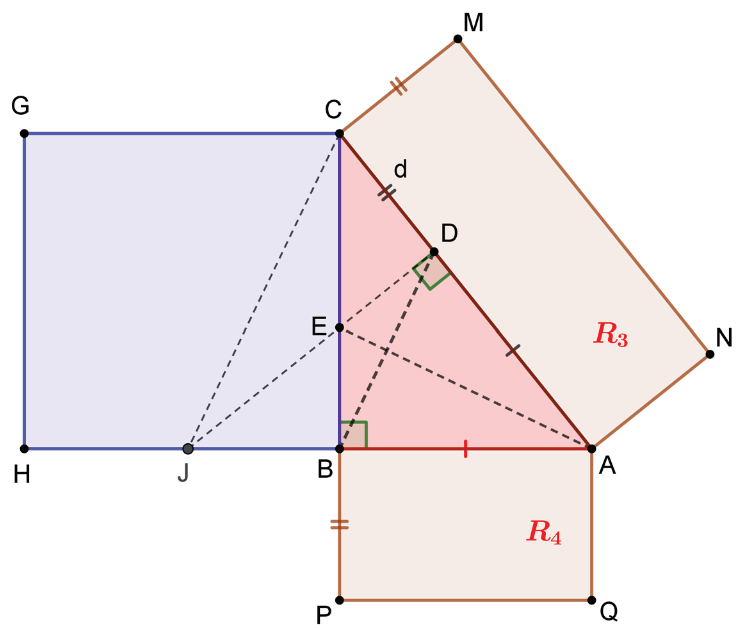


Figure 3. Proving the Right-Angled Lemma.

See Figure 2. Using the usual symbols to denote the lengths of the sides ($BC = a$, $CA = b$, $AB = c$), we must prove that $d \cdot CA + d \cdot BA = BC^2$, i.e.,

$$bd + cd = a^2. \tag{2}$$

Proof. We draw the following additional segments in the figure: AE, which bisects $\measuredangle BAC$ and is therefore perpendicular to BD; DE; and CJ parallel to DB, with J on HB. See Figure 3.

We shall prove the stated result by showing that

$$db = CE \cdot CB, \qquad dc = BE \cdot BC. \tag{3}$$

The stated result will then follow by adding these two relations, since $CE + BE = BC$.

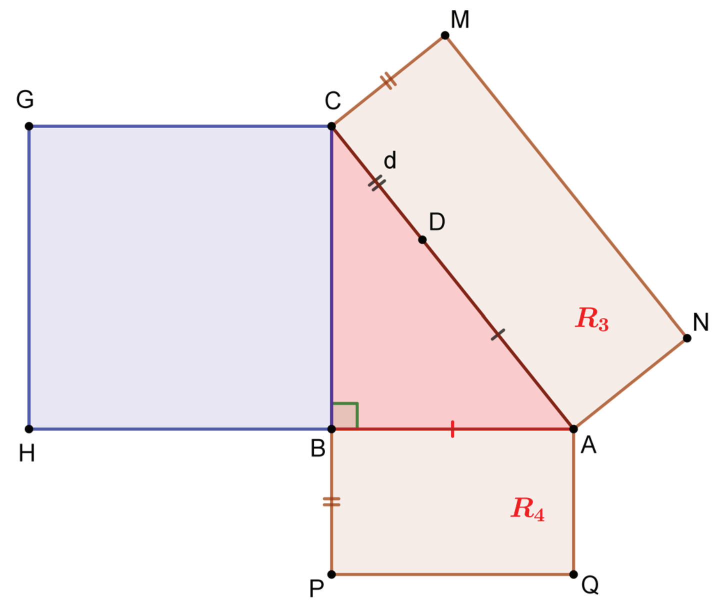


Figure 4. Proof of the PT, using the Right-Angled Lemma.

Observe that $\triangle CED \sim \triangle CAB$. Hence:

$$\frac{CD}{CE} = \frac{CB}{CA}, \qquad \therefore\ CD \cdot CA = CE \cdot CB, \qquad (4)$$

i.e., $db = CE \cdot CB$.

Next, observe that $\triangle JEB \sim \triangle CAB$. Hence:

$$\frac{EB}{JB} = \frac{AB}{CB}, \qquad \therefore\ JB \cdot AB = EB \cdot CB, \qquad (5)$$

i.e., $dc = BE \cdot BC$, since $JB = CD$ by symmetry.

So we have $db = CE \cdot CB$ and $dc = BE \cdot BC$, therefore the sum of the areas of R_3 and R_4 is equal to the area of the square, as claimed. □

Remark. As with the Isosceles Lemma, the Right-Angled Lemma is of interest in itself.

Proof of the Pythagorean theorem, based on the Right-Angled Lemma. We now show how the Right-Angled Lemma leads directly to a proof of the PT. Let $\triangle ABC$ be given, right-angled at B (see Figure 4). Using the usual symbols we must prove that $b^2 = c^2 + a^2$.

With reference to Figure 4, we have already proved that $d \cdot (b + c) = a^2$. Now by construction we have $d = b - c$. Hence:

$$\begin{aligned} (b - c) \cdot (b + c) &= a^2, \\ \therefore\ b^2 - c^2 &= a^2, \end{aligned} \qquad (6)$$

and therefore, $b^2 = c^2 + a^2$, as required. We have proved the PT! □

Remark. Basu offers the following comments to his own proof. He writes, charmingly:

"How then can one justify presenting a new and longer proof of Pythagoras' theorem? The only way to answer this is to invoke another Greek, Constantine Cavafy and his classic poem, Ithaca, which describes the long journey to Odysseus' home island. When you reach the island, the poet warns the reader, you are likely to be disappointed, for it will have little new to offer. But do not be disappointed, Cavafy tells the reader, for Ithaca's charm is the journey itself."

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