Student Corner

Problems based on the AM-GM Inequality - Part II

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In this two-part article, we consider problems from various sources which are solved using the AM-GM inequality. We list the problems first and give the solutions later. We continue from where we left off in Part I.

Problems

Problem 4. Given that x, y, z are positive real numbers and $xy + yz + zx = 1$, find the least value of

$$\frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2}. \tag{4}$$

This problem is from the *SSMJ* Problem corner, December 2014; it was proposed by Arkady Alt [5].

Problem 5. Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^2+ca} + \sqrt{b^2+bc} + \sqrt{c^2+ca} \le \sqrt{2}(a+b+c). \tag{5}$$

This problem was published in *Crux Mathematicorum*; it was proposed by Jose Luiz Diaz Barrero [6].

Keywords: AM-GM inequality

Problem 6. Let x, y, z be positive real numbers such that $x + y + z = 3$. Prove that

$$\frac{x^4 + x^2 + 1}{x^2 + x + 1} + \frac{y^4 + y^2 + 1}{y^2 + y + 1} + \frac{z^4 + z^2 + 1}{z^2 + z + 1} \geq 3xyz. \tag{6}$$

I found this problem at https://www.mat.uniroma2.it/ tauraso/AMM/AMM11815.pdf; it was proposed by G. Apostolopoulos in *American Mathematical Monthly*, [7].

Problem 7. Let a, b, c be three positive real numbers such that $ab + bc + ca = 2abc$. Prove that

$$\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \leq 2. \tag{7}$$

Solutions

Problem 4. Given that x, y, z are positive real numbers and $xy + yz + zx = 1$, find the least value of

$$\frac{x^3}{x^2 + y^2} + \frac{y^3}{y^2 + z^2} + \frac{z^3}{z^2 + x^2}.$$

Solution. From the AM-GM inequality, we have

$$\frac{x^3}{x^2 + y^2} = x - \frac{xy^2}{x^2 + y^2} \geq x - \frac{xy^2}{2xy} = x - \frac{y}{2}.$$

Similarly, we have

$$\begin{aligned} \frac{y^3}{y^2 + z^2} &\geq y - \frac{z}{2}, \\ \frac{z^3}{z^2 + x^2} &\geq z - \frac{x}{2}. \end{aligned}$$

Adding these three inequalities, we get

$$\begin{aligned} \frac{x^3}{x^2 + y^2} + \frac{y^3}{y^2 + z^2} + \frac{z^3}{z^2 + x^2} &\geq \frac{x + y + z}{2} = \frac{3}{2} \cdot \frac{x + y + z}{3} \\ &\geq \frac{3}{2} \cdot \left(\frac{xy + yz + zx}{3} \right)^{1/2} = \frac{\sqrt{3}}{2}. \end{aligned}$$

Hence the minimum value of $\frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2}$ is $\frac{1}{2}\sqrt{3}$.

Problem 5. Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^2 + ca} + \sqrt{b^2 + bc} + \sqrt{c^2 + ca} \leq \sqrt{2}(a + b + c).$$

Solution. We have:

$$\begin{aligned}
\sqrt{a^2+ca}+\sqrt{b^2+bc}+\sqrt{c^2+ca} &= \frac{1}{\sqrt{2}}\cdot\left(\sqrt{2a\cdot(a+c)}+\sqrt{2b\cdot(b+c)}+\sqrt{2c\cdot(c+a)}\right)\\
&\le \frac{1}{\sqrt{2}}\cdot\left(\frac{2a+(a+c)}{2}+\frac{2b+(b+c)}{2}+\frac{2c+(c+a)}{2}\right)\\
&= \frac{1}{2\cdot\sqrt{2}}\cdot(4a+4b+4c)=\sqrt{2}\cdot(a+b+c).
\end{aligned}$$

Problem 6. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1}\ge 3xyz.$$

Solution. We have

$$\begin{aligned}
\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1} &= \left(x^2-x+1\right)+\left(y^2-y+1\right)+\left(z^2-z+1\right)\\
&= 3\cdot\frac{x^2+y^2+z^2}{3}\ge 3\cdot\left(\frac{x+y+z}{3}\right)^2=3.
\end{aligned}$$

Also:

$$3=3\cdot\left(\frac{x+y+z}{3}\right)^3\ge 3xyz,$$

hence proved.

Problem 7. Let a, b, c be three positive real numbers such that $ab+bc+ca=2abc$. Prove that

$$\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\le 2.$$

Solution (published in 2019 in the issue 1 of AMJ journal). From $ab+bc+ca=2abc$ we get

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2.$$

Next, using the AM-GM inequality, we get

$$\begin{aligned}
\frac{1}{a}+\frac{1}{b}&\ge\frac{2}{\sqrt{ab}},\\
\frac{1}{b}+\frac{1}{c}&\ge\frac{2}{\sqrt{bc}},\\
\frac{1}{c}+\frac{1}{a}&\ge\frac{2}{\sqrt{ca}}.
\end{aligned}$$

By addition we get

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}},$$

and the stated result follows.

References

1. Florin Rotaru, *Mathematical Reflections* 2009, J468-Solution, Issue-1, https://www.awesomemath.org/wp-pdf-files/math-reflections/mr-2019-01/mr_6_2018_solutions_2.pdf
2. Mihaela Berindeanu, "Problem EM-55", *Arhimede math.*, j. 5.1 (2018), pg. 33
3. Toyesh Prakash Sharma, "Generalization of Problem E 55", *Arhimede math.*, 7.2 (2018), pg. 136
4. Albert Stadler, "Prob. 5303, Angel Plaza", SSMJ problem solution corner, Nov. 2014, pg. 7-9

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