Maxima and Minima using the Power Mean Inequality

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Student Corner

The power mean inequality states the following:

For positive quantities $a_1, a_2, \cdots, a_n$ *and* $p \geq 1$,

$$\frac{{a_1}^p + {a_2}^p + \cdots + {a_n}^p}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p.$$

For $0 \leq p \leq 1$, *inequality reverses.*

Proof

We know that for $p \geq 1, f(x) = x^p$ is convex; the graph has the following appearance (Figure 1):

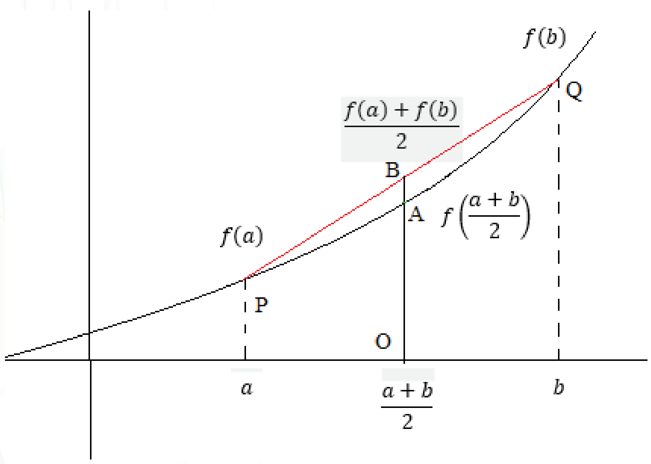

Figure 1

From Figure 1 we have

$$OB \geq OA,$$

$$\Rightarrow \frac{f(a) + f(b)}{2} \geq f\left(\frac{a+b}{2}\right).$$

Here $f(a) = a^p, f(b) = b^p$ and $f\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2}\right)^p$. Hence

$$\frac{a^p + b^p}{2} \geq \left(\frac{a+b}{2}\right)^p.$$

This is the power mean inequality for two variables. Now let $a = \frac{a_1 + a_2}{2}$ and $b = \frac{b_1 + b_2}{2}$. Then

$$\frac{\left(\frac{a_1+a_2}{2}\right)^p + \left(\frac{b_1+b_2}{2}\right)^p}{2} \geq \left(\frac{\frac{a_1+a_2}{2} + \frac{b_1+b_2}{2}}{2}\right)^p.$$

Using the power mean inequality for $\left(\frac{a_1+a_2}{2}\right)^p$ and $\left(\frac{b_1+b_2}{2}\right)^p$ we get:

$$\frac{\frac{{a_1}^p + {a_2}^p}{2} + \frac{{b_1}^p + {b_2}^p}{2}}{2} \geq \frac{\left(\frac{a_1+a_2}{2}\right)^p + \left(\frac{b_1+b_2}{2}\right)^p}{2}.$$

So,

$$\frac{\frac{{a_1}^p + {a_2}^p}{2} + \frac{{b_1}^p + {b_2}^p}{2}}{2} \geq \left(\frac{\frac{a_1+a_2}{2} + \frac{b_1+b_2}{2}}{2}\right)^p,$$

$$\Rightarrow \frac{{a_1}^p + {a_2}^p + {b_1}^p + {b_2}^p}{4} \geq \left(\frac{a_1 + a_2 + b_1 + b_2}{4}\right)^p.$$

This is power mean inequality for four variables. Similarly for n quantities we have:

$$\frac{{a_1}^p + {a_2}^p + \cdots + {a_n}^p}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p ; p \geq 1.$$

For $0 \leq p \leq 1$ the same function i.e., $f(x) = x^p$ is concave function, so $\frac{f(a) + f(b)}{2} \leq f\left(\frac{a+b}{2}\right)$ and

$$\frac{{a_1}^p + {a_2}^p + \cdots + {a_n}^p}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p ; 0 \leq p \leq 1.$$

Problem 1. Find the maximum and minimum values of the function $\sin x + \cos x$.

Solution

The power mean inequality states that for real positive quantities a, b and $p \geq 1$,

$$\frac{a^p + b^p}{2} \geq \left(\frac{a+b}{2}\right)^p.$$

Letting $a = \sin x$ and $b = \cos x$ and $p = 2$, we get

$$\frac{\sin^2 x + \cos^2 x}{2} \geq \left(\frac{\sin x + \cos x}{2}\right)^2 \Rightarrow \frac{1}{2} \geq \left(\frac{\sin x + \cos x}{2}\right)^2$$

$$\Rightarrow 2 \geq (\sin x + \cos x)^2 = \begin{cases} \sin x + \cos x \geq -\sqrt{2} \\ \sin x + \cos x \leq \sqrt{2} \end{cases}$$

$\therefore \sqrt{2}$ is the maximum value of $\sin x + \cos x$ and $-\sqrt{2}$ is the minimum value of $\sin x + \cos x$.

The above problem is given as an exercise in a chapter of class 12 entitled as "Application of Derivatives" [1].

Problem 2. For $a, n \geq 1$, find the maximum value of $\sqrt[n]{\sin ax} + \sqrt[n]{\cos ax}$.

(Note: This is meant to be done without using calculus.)

Solution

The power mean inequality states for real quantities u, v and $p \geq 1$,

$$\frac{u^{1/p} + v^{1/\mathrm{p}}}{2} \leq \left(\frac{u+v}{2}\right)^{1/\mathrm{p}}.$$

Letting $u = \sin^2 ax$, $v = \cos^2 ax$ and $p = 2n$ gives

$$\frac{\left(\sin^2 ax\right)^{1/2n} + \left(\cos^2 ax\right)^{1/2\mathrm{n}}}{2} \leq \left(\frac{\sin^2 ax + \cos^2 ax}{2}\right)^{1/2\mathrm{n}},$$

$$\text{hence: } \sqrt[n]{\sin ax} + \sqrt[n]{\cos ax} \leq 2\left(\frac{1}{2}\right)^{\frac{1}{2n}} = 2^{1-1/2\mathrm{n}}.$$

Hence the maximum value of the function is $2^{1-1/2n}$.

The next problem was proposed by Jose L.D-Barrero [2]

Problem 3. Let a, b, c, d be four positive real numbers. Find the minimum value of

$$\frac{\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c} + \sqrt[4]{d}}{\sqrt[4]{a+b+c+d}}.$$

Solution

From the power mean inequality:

$$\frac{\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c} + \sqrt[4]{d}}{4} \leq \left(\frac{a+b+c+d}{4}\right)^{\frac{1}{4}},$$

$$\Rightarrow \frac{\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c} + \sqrt[4]{d}}{\sqrt[4]{a+b+c+d}} \leq 4\left(\frac{1}{4}\right)^{\frac{1}{4}} = 2\sqrt{2}$$

Thus, the maximum value of $\frac{\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c} + \sqrt[4]{d}}{\sqrt[4]{a+b+c+d}}$ is $2\sqrt{2}$ which is attained when $a = b = c = d$.

References

1. Problem 3 (iii), Exercise 6.5, Chapter 6, "Applications of Derivatives" in Textbook of mathematics Class-12, NCERT, p.232. ISBN-978-8174506290.
2. Jose L.D-Barrero, Problem 857, The Problem Corner, The Pentagon, Vol. 80 No. 1 Fall 2020. P. 29.

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