Geometric and Calculus Proofs of Some Inequalities

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Introduction

There are many visual and calculus-based proofs of $\pi^e < e^\pi$ and $b^a < a^b$(for $e \le a < b$). The aim of this paper is to give geometric and calculus-based proofs of these inequalities. We show in addition that $b^a > a^b$ for $0 < a < b \le e$ and $a^b \le 1 < b^a$ for $0 < a \le 1 < b$.

To show $\pi^e < e^\pi$, Nelson ([2]) uses the fact that the curve $y = e^{x/e}$ lies above the line $y = x$, while Nakhli ([1]) uses the fact that the curve $y = \dfrac{\ln x}{x}$ attains the global maximum at the point e.

Proofs using the curve $y = \frac{1}{x}$

Here we use the curve $y = \frac{1}{x}$ and the fact that the shaded region in Figure 1 lies within the rectangle bounded by the lines $x = \frac{1}{\pi}$, $x = \frac{1}{e}$, the x-axis and the line $y = \pi$.

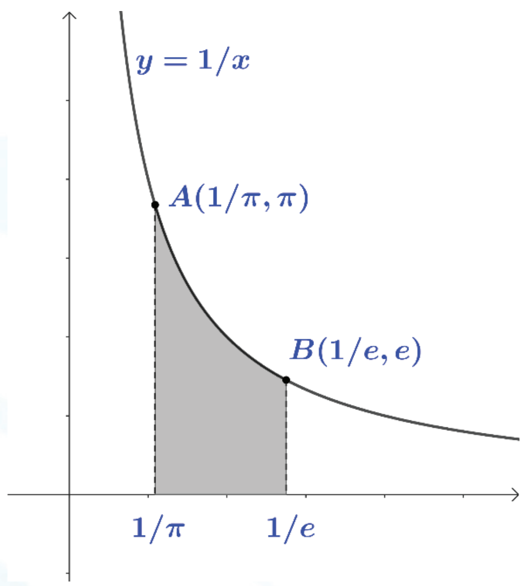

Figure 1. Geometric visualisation of $\pi^e < e^\pi$

Keywords: Proofs, graphs, calculus, inequalities.

From Figure 1, as $e < \pi \Rightarrow \frac{1}{\pi} < \frac{1}{e}$, we have

$$\ln\left(\frac{1}{e}\right) - \ln\left(\frac{1}{\pi}\right) = \int_{\frac{1}{\pi}}^{\frac{1}{e}} \frac{1}{x} dx < \pi\left(\frac{1}{e} - \frac{1}{\pi}\right),$$

$$\Rightarrow -\ln e + \ln \pi < \frac{\pi - e}{e} \Rightarrow \ln \pi - 1 < \frac{\pi}{e} - 1 \Rightarrow \ln \pi < \frac{\pi}{e} \Rightarrow \pi^e < e^\pi$$

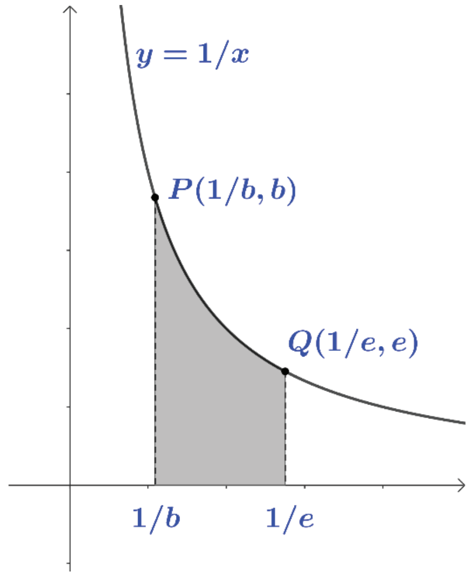

Figure 2. Geometric visualization of $b^e < e^b$ for $e < b$

From Figure 2, as $e < b \Rightarrow \frac{1}{b} < \frac{1}{e}$, we have

$$\ln\left(\frac{1}{e}\right) - \ln\left(\frac{1}{b}\right) = \int_{1/b}^{1/e} \frac{1}{x} dx < b\left(\frac{1}{e} - \frac{1}{b}\right)$$

$$\Rightarrow -\ln \mathrm{e} + \ln b < \frac{b - e}{e} \Rightarrow \ln b - 1 < \frac{b}{e} - 1 \Rightarrow \ln b < \frac{b}{e} \Rightarrow b^e < e^b$$

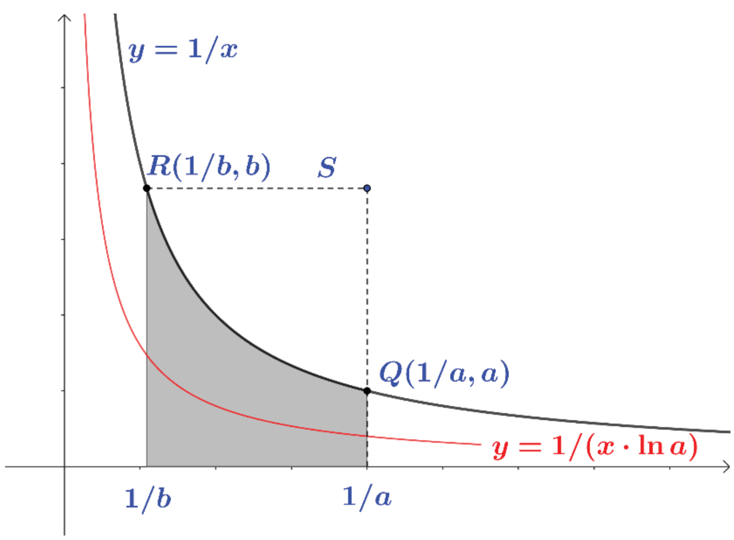

Figure 3. Geometric visualization of $b^a < a^b$ for $e \leq a < b$

From Figure 3, as $e \le a < b \Rightarrow \frac{1}{b} < \frac{1}{a} \le \frac{1}{e}$, we have

$$\begin{aligned} \frac{\ln(1/a)}{\ln a} - \frac{\ln(1/b)}{\ln a} &= \int_{1/b}^{1/a} \left(\frac{1}{x\ln a}\right) dx < b\left(\frac{1}{a} - \frac{1}{b}\right), \\ \Rightarrow \frac{-\ln a}{\ln a} + \frac{\ln b}{\ln a} &< \frac{b-a}{a} \Rightarrow \frac{\ln b}{\ln a} - 1 < \frac{b}{a} - 1 \\ \Rightarrow \frac{\ln b}{\ln a} &< \frac{b}{a} \Rightarrow a\ln b < b\ln a \Rightarrow b^a < a^b \end{aligned}$$

Proof using calculus

Consider the function $f(x) = \left(\frac{1}{x}\right)^x$. Taking natural logarithms on both sides we get,

$\ln f(x) = x\ln\left(\frac{1}{x}\right) = -x \cdot \ln x$

$\Rightarrow \frac{f'(x)}{f(x)} = -\ln x - 1 = -(1 + \ln x)$

$\Rightarrow f'(x) = -f(x)(1 + \ln x)$.

Now, if $\ln x + 1 < 0 \Rightarrow \ln x < -1 \Rightarrow x < e^{-1} \Rightarrow x < \frac{1}{e}$.

Similarly, if $\ln x + 1 > 0 \Rightarrow x > \frac{1}{e}$ and $\ln x + 1 = 0 \Rightarrow x = \frac{1}{e}$.

Therefore, $x \in \left(0, \frac{1}{e}\right] \Rightarrow f'(x) \ge 0 \Rightarrow f(x)$ is strictly increasing in $\left(0, \frac{1}{e}\right)$

Similarly, $x \in \left[\frac{1}{e}, \infty\right) \Rightarrow f'(x) \le 0 \Rightarrow f(x)$ is strictly decreasing in $\left(\frac{1}{e}, \infty\right)$.

Case 1. $e \le a < b$

Now, $e \le a < b \Rightarrow \frac{1}{b} < \frac{1}{a} \le \frac{1}{e} \Rightarrow \left(\frac{1}{1/b}\right)^{1/b} < \left(\frac{1}{1/a}\right)^{1/a}$

$\left(\text{since } f(x) \text{ is strictly increasing in } \left(0, \frac{1}{e}\right)\right)$

$\Rightarrow b^{1/b} < a^{1/a} \Rightarrow b^a < a^b$.

Case 2. $0 < a < b \le e$

Now, $0 < a < b \le e \Rightarrow \frac{1}{e} \le \frac{1}{b} < \frac{1}{a} < \infty \Rightarrow \left(\frac{1}{1/b}\right)^{1/b} > \left(\frac{1}{1/a}\right)^{1/a}$

$\left(\text{since } f(x) \text{ is strictly decreasing in } \left(\frac{1}{e}, \infty\right)\right)$

$\Rightarrow b^{1/b} > a^{1/a} \Rightarrow b^a > a^b$.

Case 3. $0 < a \le 1 < b$

Now, $0 < a \le 1 < b \Rightarrow \frac{1}{b} < 1 \le \frac{1}{a} < \infty \Rightarrow \ln\left(\frac{1}{b}\right) < \ln 1 \le \ln\left(\frac{1}{a}\right) \Rightarrow \ln\left(\frac{1}{b}\right) < 0 \le \ln\left(\frac{1}{a}\right)$

$\Rightarrow -\ln b < 0$ and $0 \le -\ln a \Rightarrow \ln b > 0$ and $0 \ge \ln a$

$\Rightarrow a\ln b > 0$, and $0 \ge b\ln a$

$\Rightarrow b^a > 1$ and $a^b \le 1 \Rightarrow a^b \le 1 < b^a$.

Remark 1. $\pi^e < e^\pi$

As $e < \pi$, by Case 1, we have $\pi^e < e^\pi$.

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References

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