Centroids of Quadrilaterals and a Peculiarity of Parallelograms

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Dedicated to Arnold Kirsch (Germany, 1922-2013) on the 10th anniversary of his death.

Abstract: We analyze briefly different kinds of centroids of quadrilaterals and give geometrical and elementary proofs that in the world of quadrilaterals, *only parallelograms* have the property that their *laminar centroid* coincides with the *vertex centroid*. This paper is based on short papers (in German) by Arnold Kirsch (Kassel, Germany, 1922-2013) published between 1987 and 1995. We think these deserve to be better known – published proofs in mathematical journals in English language are usually rather complex (e.g., Kim 2016, 2020).

Definitions. A *lamina* is a flat object of uniform thickness. The *laminar centroid* of a flat region is the centre of gravity of the region when it is regarded as a thin lamina. It is also called the *geometric centroid*. The *vertex centroid* of a polygon is the centre of gravity of a system of unit masses placed at the vertices of the polygon.

In the world of triangles, the *laminar centroid* always coincides with the *vertex centroid* (intersection point of the medians). This is an elementary and well-known fact. We proceed to prove this using the principle of levers from elementary physics.

Keywords: Geometry, centroids, parallelograms, laminar centroid, vertex centroid

Lemma 1: The vertex centroid G of a pair of point masses (weights $w_1$and w_2) lies on the connecting line and the corresponding distances l_1, l_2 have the ratio $\frac{l_1}{l_2} = \frac{w_2}{w_1}$. For mechanical purposes, one can imagine that at the centroid G, a combined weight $w_1 + w_2$ is concentrated. In terms of analytical geometry, the point G is the *weighted arithmetic mean* of the points G_1 and G_2:

$$G = \frac{w_1}{w_1 + w_2} \cdot G_1 + \frac{w_2}{w_1 + w_2} \cdot G_2.$$

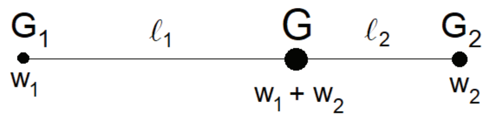

Figure 1. Law of levers.

Drawing on this fact, one can give a physically motivated proof that the medians of a triangle concur, intersecting each other in the ratio 2: 1. Assume that at the vertices of a triangle we have unit point masses, and we want to determine the centroid of these three point masses (we call this the 'vertex centroid'). Lemma 1 tells us that the centroid of the pair of unit masses at A and B is at the midpoint M_{AB} of AB, where we then have mass 2. Using Lemma 1 again, we see that the centroid of the three unit-masses lies on the median $m_c = CM_{AB}$, at the point G such that CG: $GM_{AB} = 2$: 1. We may imagine all three unit-masses to be concentrated at G (with total mass 3).

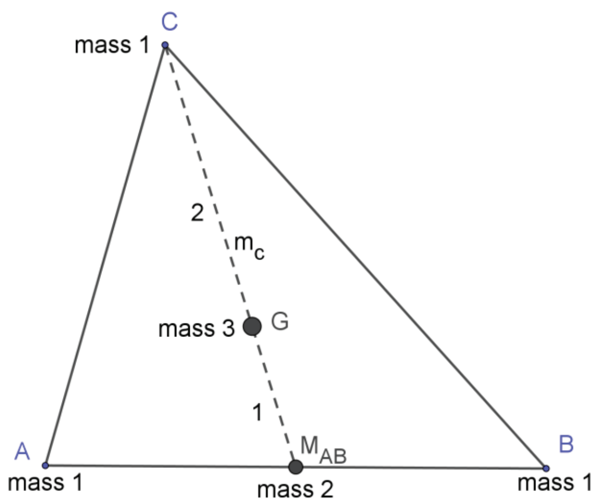

Figure 2. The triangle centroid as the *vertex centroid.*

Since the same must hold for the other two medians, and the centroid is unique, we have proven two things: (1) The medians concur at a point that trisects all three of them; and (2) the point of concurrence is the *vertex centroid*. Note that this approach does not explain why the laminar centroid of the triangle lies at the same point. Here is one approach which explains why. We divide the triangle into infinitesimally thin stripes parallel to AB. Each of these stripes has its center of mass at its midpoint, so the center of mass of the whole lamina must lie somewhere on the line consisting of all these midpoints, which is the median CM_{AB}. By a symmetric argument, it must also lie on the other two medians, hence the intersection point of the three medians is also the laminar centroid.

Let us denote the *laminar centroid* of a polygon by G_L and its *vertex centroid* by G_V.

The following must be noted. The property $G_L = G_V$ is a peculiarity of triangles (in the sense that it is true for *all* triangles), but for other polygons this is not necessarily true. Of course, for regular polygons $G_L = G_V$ still holds (by symmetry, both must lie at the centre of the polygon), but for general polygons it is of great interest to ask: For which polygons is it true that $G_L = G_V$? We will restrict our exploration in this only article to *quadrilaterals* and ask: Which quadrilaterals have the property $G_L = G_V$? (We are not aware if there are any results of this kind for polygons with more than 4 vertices.)

It is easy to see that *parallelograms* have the property that $G_L = G_V$ (= intersection point of the two diagonals). Assume we have a parallelogram *ABCD* with unit mass at each vertex. Then according to Lemma 1, the intersection point of the diagonals (where they bisect each other) is the centroid of the two masses at *A* and *C* (mass 2 units), and also of the two masses at *B* and *D* (again mass 2 units). Hence the point of intersection of the diagonals is the vertex centroid G_V.

To see that this is G_L too, we consider the laminar centroids of triangles *ABC* and *ADC*, namely, $G_L(ABC)$ and $G_L(ADC)$. (These coincide with respective vertex centroids.) Both lie on the diagonal *BC* and lie at equal distance from the point of intersection of the diagonals (note the half-turn symmetry of a parallelogram with this point as centre). For mechanical purposes we can imagine the whole masses of triangles *ABC* and *ADC* being concentrated at these two triangle laminar centroids. And since these masses (areas) are equal, it follows that the laminar centroid of the whole parallelogram lies at the intersection point of the diagonals (Figure 3). So, all parallelograms have the property that $G_L = G_V$.

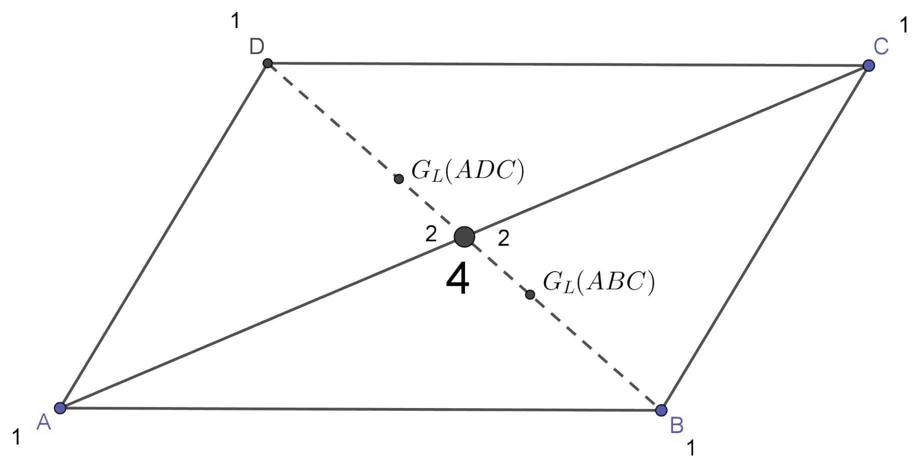

Figure 3. The intersection point of the diagonals of a parallelogram are G_V and G_L.

What has been proved above is well known. Now for the not so well-known part: *parallelograms are the only quadrilaterals with the property $G_L = G_V$*. Here Arnold Kirsch (of Germany, a very deserving professor for mathematics and mathematics education at the University of Kassel) came up with a very elementary proof which students in grades 9 or 10 can follow and which not only answers the question (*verification*) but explains *why* it is true (*explanation*). *Verification* and *explanation* are two important functions of proof (but there are also others, see De Villiers 2012).

We did not find an elementary proof in the English literature (if somebody happens to know one, please inform the author), hence we wanted to share his ingenious ideas, formulated in German (Kirsch 1987, 1995), with potentially more readers in the English language.

Theorem: A quadrilateral has the property that $G_L = G_V$ if and only if it is a parallelogram.

For this topic, we can omit crossed quadrilaterals because it is not so clear what is the interior of such a quadrilateral, and we restrict to convex or concave quadrilaterals (Figure 4a, 4b). In both cases there is an interior diagonal (AC in Fig. 4) which itself or its extension meets the other diagonal (BD in Fig. 4).

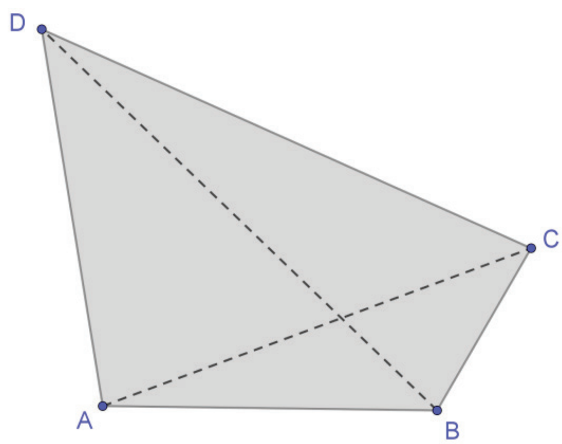

Figure 4a. Convex quadrilateral.

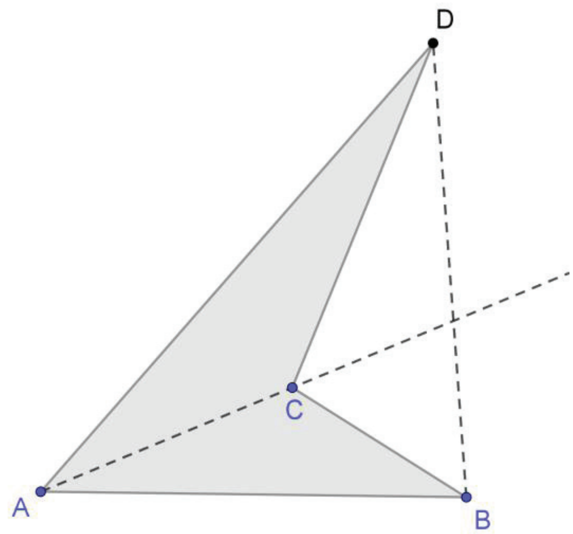

Figure 4b. Concave quadrilateral.

The part "if" of the Theorem (the easy and well-known part) has been dealt with above. For the "only if" part we use another lemma. We lay the groundwork for this by stating two facts:

(a) If the vertex D of ΔDAC is moved parallel to AC by $\boldsymbol{u}$, then the centroid G_2 of ΔDAC is moved by $\frac{1}{3}\boldsymbol{u}$ (see Figure 5; here M denotes the midpoint of AC). This should be clear since $G_2 = \frac{D+A+C}{3}$.

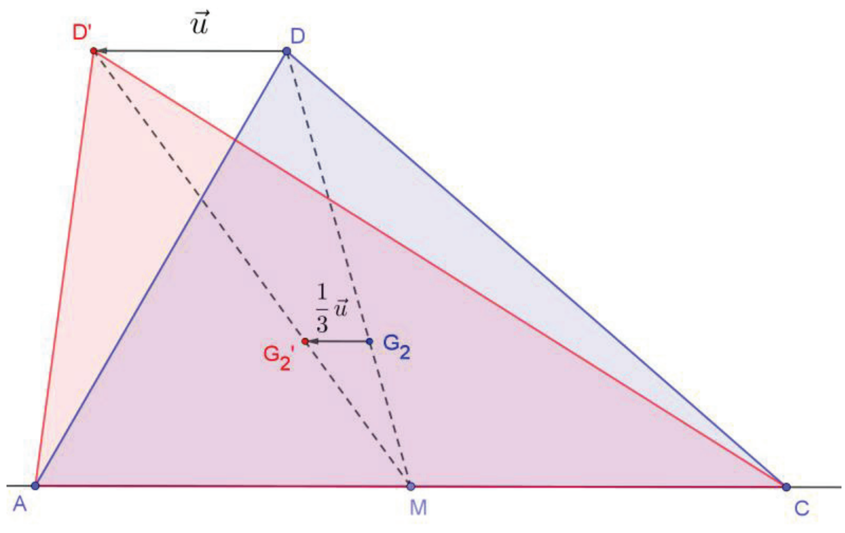

Figure 5. If $D' = D + \boldsymbol{u}$, then $G_2' = G_2 + \frac{1}{3}\boldsymbol{u}$.

(b) If vertices A and C of ΔABC are moved along the straight line AC by $\boldsymbol{v}$, then the centroid G_1 of $\Delta ABC$is moved by $\frac{2}{3}\boldsymbol{v}$ (see Figure 6; M is the midpoint of AC; M' is the midpoint of $A'C'$). This is so since $G_1 = \frac{A+B+C}{3}$.

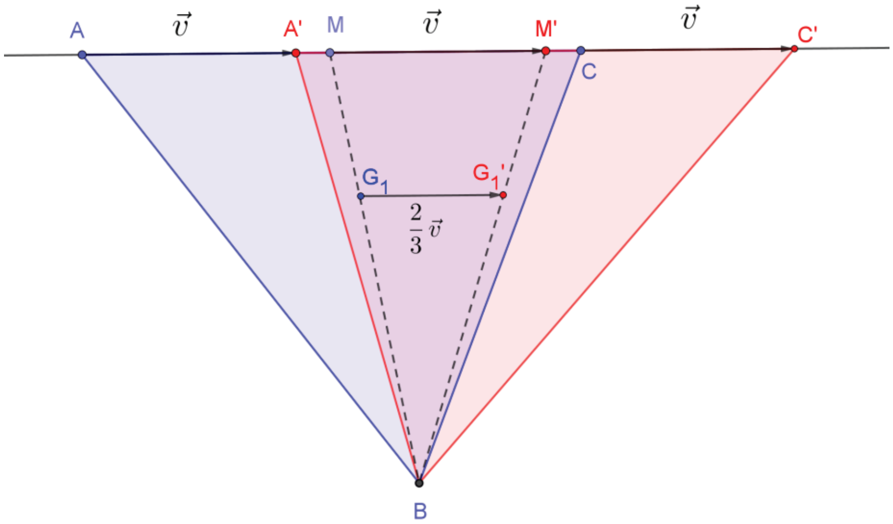

Figure 6. If $A' = A + \boldsymbol{v}$ and $C' = C + \boldsymbol{v}$, then $G_1' = G_1 + \frac{2}{3}\boldsymbol{v}$.

These two facts and the following lemma may also be formulated using idea of *shear mappings*, but this is probably not very well-known at school. Knowledge of shear mappings is not necessary; we can do without it (the *intercept theorem* or *homothety* suffice).

Lemma 2: Let $ABCD$ be a quadrilateral with interior diagonal AC. Let the points A, C be translated along AC by a vector $\boldsymbol{v}$ to A', C'; let the points B, D be translated by $-\boldsymbol{v}$ to B', D'. Then quadrilateral $A'B'C'D'$ has the same vertex centroid as quadrilateral $ABCD$, and the laminar centroid G_L of $ABCD$ maps to the laminar centroid G_L' of $A'B'C'D'$ via the translation $\frac{1}{3}\boldsymbol{v}$ (see Figure 7).

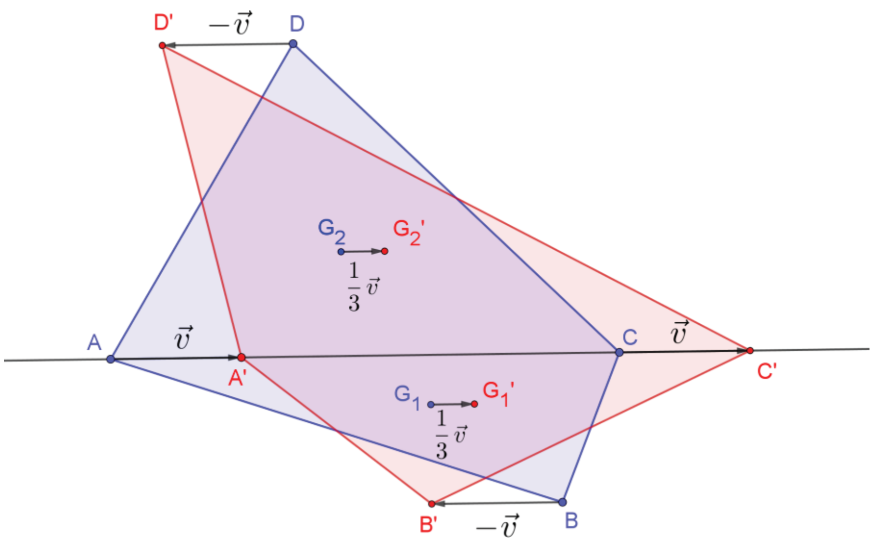

Figure 7. To Lemma 2

Proof of Lemma 2: The first claim in Lemma 2 is immediately clear because the total shift of all four points together is **0**.

From the facts **a)** and **b)** presented earlier, the shifts $G_1 \longmapsto G_1'$ and $G_2 \longmapsto G_2'$ of the centroids of the triangles are given by $\frac{1}{3}\boldsymbol{v}$. Now by Lemma 1, the laminar centroid G_L of $ABCD$ divides line segment G_1G_2 in the same ratio as the laminar centroid G_L' of $A'B'C'D'$ divides line segment $G_1'G_2'$. This ratio is given by the *areas* of the two triangles; these are the weights. But since the areas of the triangles do not change (same base and same altitude), the ratios are the same! It follows that the shift $G_L \longrightarrow G_L'$ is given by $\frac{1}{3}\boldsymbol{v}$, too. ■

Now we are ready to **prove** the 'only if' part of the **Theorem**. Let $ABCD$ be a quadrilateral with interior diagonal AC and the property $G_L = G_V$. We want to prove that it must be a parallelogram. We apply the operation of Lemma 2 to $ABCD$, choosing the vector $\boldsymbol{v}$ along AC in such a way that the diagonal $A'C'$ of the new quadrilateral is *bisected* by the other diagonal $B'D'$ at points $M' = N'$ (for this, we choose $\boldsymbol{v} = \frac{1}{2}\boldsymbol{MN}$ where N denotes the intersection point of AC and BD; see Figure 8).

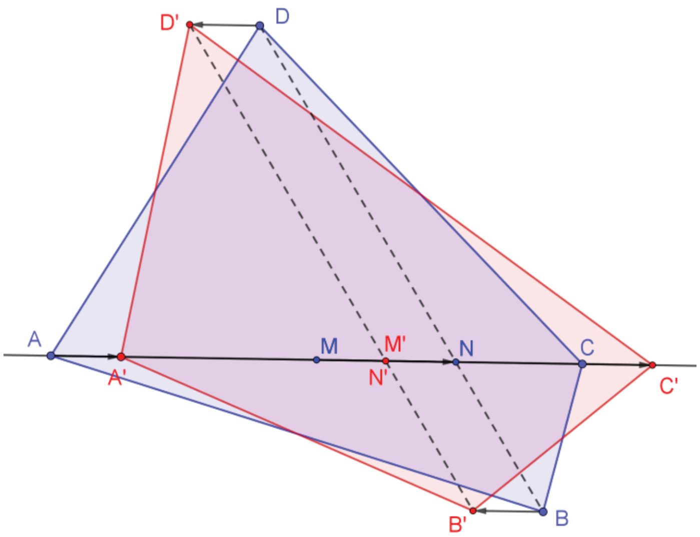

Figure 8. The operation of Lemma 2 with $\boldsymbol{v} = \frac{1}{2}MN$.

After this operation $B'D'$ will surely be an interior diagonal of the quadrilateral $A'B'C'D'$, and again we apply the operation of Lemma 2, this second time with vector $\boldsymbol{w}$ parallel to $B'D'$, and we choose $\boldsymbol{w}$ in such a way that the diagonal $B''D''$ of the image quadrilateral $A''B''C''D''$is bisected by the other diagonal $A''C''$. Now both diagonals bisect each other, which means that $A''B''C''D''$ must be a parallelogram. We know that in parallelograms $G_L = G_V$ holds, so

$$G_L\left(A''B''C''D''\right) = G_V\left(A''B''C''D''\right).$$

Since the vertex centroid did not change when applying the two operations, we know that

$$G_V\left(A''B''C''D''\right) = G_V\left(ABCD\right),$$

hence $G_L\left(A''B''C''D''\right) = G_V\left(ABCD\right)$.

On the other hand, we know that

$$G_L\left(A''B''C''D''\right) = G_L\left(ABCD\right) + \frac{1}{3}\boldsymbol{v} + \boldsymbol{w},$$

and since $G_V(ABCD) = G_L(ABCD)$, this means that this added shift vector $\frac{1}{3}\boldsymbol{v} + \boldsymbol{w}$ must vanish. This vanishes only for $\boldsymbol{v} = \boldsymbol{0} = \boldsymbol{w}$, which means that $ABCD$ is a parallelogram. ■

This was, roughly spoken (we made some additional sketches and did not translate literally), the version of Kirsch 1987. Then K. Seebach (Munich) came up with another purely geometric proof (Seebach 1994) using the principle of *homothety*. And it was again A. Kirsch, 1995, who made this proof still easier and shorter, and he used the words (translated from German) "hereby probably the ideal geometric proof of the statement is found!"

This proof needs the knowledge how to construct the *laminar centroid* of a quadrilateral $ABCD$. First, draw the diagonal AC and the laminar centroids of ΔABC and ΔADC, i.e., $G_L(ABC)$ and $G_L(ADC)$. The laminar centroid $G_L(ABCD)$ of the whole quadrilateral must lie on the line segment connecting $G_L(ABC)$ and $G_L(ADC)$ (we even know where, but now that is not important).

Doing the same with the other diagonal BD, we know that $G_L(ABCD)$ must be the point of intersection of the segments $G_L(ABC)\ G_L(ADC$ and $G_L(ABD)\ G_L(CBD)$. One also must know how to construct the vertex centroid of a quadrilateral $ABCD$: it is the midpoint of the line segment joining the midpoints of the two diagonals. These two principles were already used in Figure 3.

Assume that $ABCD$ is not a parallelogram. Let S be the intersection point of the diagonals (Figure 9); then S is not simultaneously the midpoint of both the diagonals.

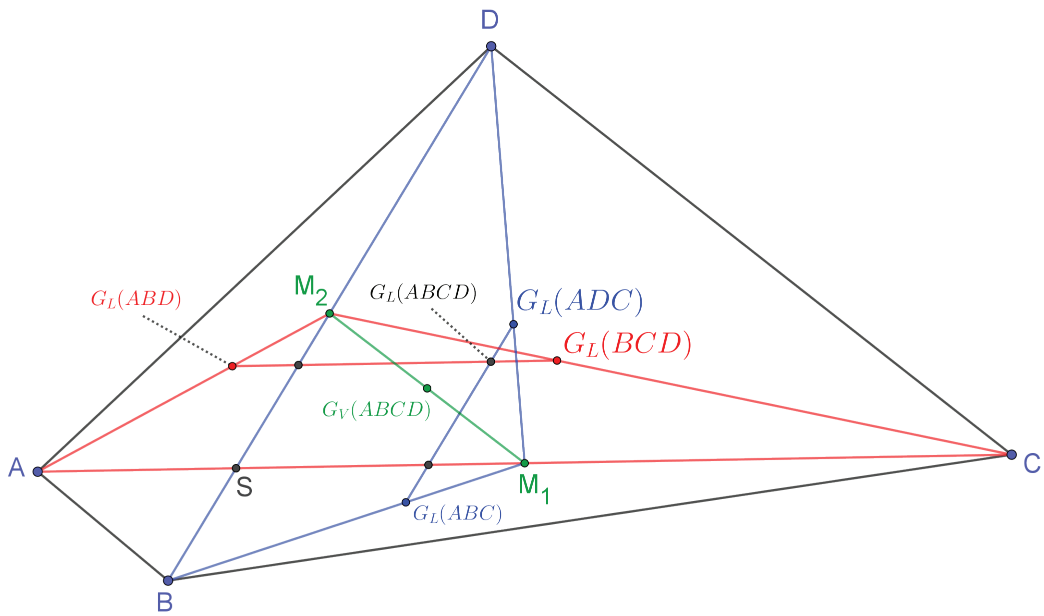

Figure 9. Very short and purely geometric proof – *convex* case

Then, using the well-known properties of triangle centroids, the intercept theorem, and its converse, one can see immediately (note the parallelogram with opposite vertices S and $G_L(ABCD)$:

$$\boldsymbol{SG_L}\,(ABCD) = \frac{2}{3}\boldsymbol{SM_1} + \frac{2}{3}\boldsymbol{SM_2} = \frac{2}{3}\,(\boldsymbol{SM_1} + \boldsymbol{SM_2}) \neq \frac{1}{2}\,(\boldsymbol{SM_1} + \boldsymbol{SM_2}) = \boldsymbol{SG_V}\,(ABCD)\,, \qquad (*)$$

hence $G_L\,(ABCD) \neq G_V(ABCD)$.

Remarks

- Note that our precondition is that while we do not have $M_1 = S = M_2$(because $ABCD$ is not a parallelogram), the case $M_1 = S \neq M_2$ is covered by the above.
- The use of vector notation in (*) is just for abbreviation; one could easily avoid it and describe with more words the resulting parallelogram with opposite vertices S and $G_L(ABCD)$. Thus, this proof can be seen as purely geometric, and not analytic, although we used vectors in (*).

In the concave case nearly nothing changes (Figure 10), the only difference is that S lies in the exterior of $ABCD$ and the two triangles ΔABD and ΔCBD are not "addeD" (for getting the quadrilateral $ABCD$) but "subtracteD".

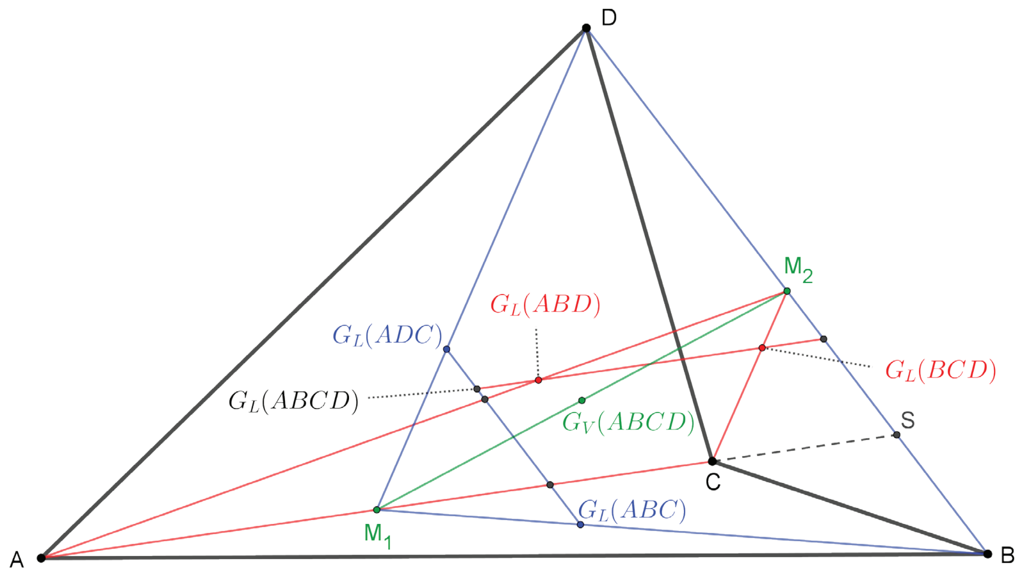

Figure 10. Short and purely geometric proof – *concave* case

Here is another short proof of the **Theorem** with *coordinates*, *vectors,* and an *oblique coordinate system*.

We use an oblique coordinate system. The origin lies in the intersection point of the diagonals of the quadrilateral. The first axis is the straight line AC and the second BD. Then the vertices are:

$A = (a, 0)$;	$B = (0, b), b < 0$;	$C = (c, 0), c > a$;	$D = (0, d), d > 0$.

Then the *vertex centroid* is given by

$$G_V = \left(\frac{a+c}{4}, \frac{b+d}{4}\right).$$

The centroid of the ΔABC is

$$G_1 = \left(\frac{a+c}{3}, \frac{b}{3}\right),$$

and the centroid of ΔADC is

$$G_2 = \left(\frac{a+c}{3}, \frac{d}{3}\right).$$

According to Lemma 1, the laminar centroid G_L of the quadrilateral $ABCD$ is the *weighted mean* of the points G_1 and G_2, where the weights are the triangle areas or weights proportional to these areas, namely, $-b$ and d:

$$G_L = \frac{-b}{(-b)+d} \cdot \left(\frac{a+c}{3}, \frac{b}{3}\right) + \frac{d}{(-b)+d} \cdot \left(\frac{a+c}{3}, \frac{d}{3}\right) = \left(\frac{a+c}{3}, \frac{b+d}{3}\right).$$

Hence, $G_L = G_V$ holds if and only if $a = -c$ and $b = -d$, i.e., $ABCD$ is a parallelogram. ■

Conclusion

In many cases school students get wrong impressions concerning centroids (e.g., that there is only one kind of centroid, or that if distinguished at all, the *laminar centroid* necessarily coincides with the *vertex centroid*, as with triangles). Dealing with that topic in case of quadrilaterals (how to determine the laminar centroid of a quadrilateral, parallelograms have the property $G_L = G_V$, and only they have this property, and so on) provides a possible way to prevent this misconception. Many proofs for "only parallelograms have this property" are too complicated to be treated at school but Kirsch's proofs are elementary, purely geometric and students can easily follow every single step. Of course, it cannot be expected that students find these steps on their own; this was an ingenious idea of Arnold Kirsch. And using the alternative proof using *analytic geometry* provides a good opportunity to make use of oblique coordinate systems.

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