Opening Bracket . . .

On Proof and Reasoning in the Teaching of Mathematics

Proof is central to the discipline of mathematics, and the exercise of looking for new proofs for known results, or proofs that embody an aesthetic element, is highly valued by mathematicians. G H Hardy writes in *A Mathematician's Apology* (which has been reviewed in this issue) about a method of proof that is very dear to mathematicians – 'proof by contradiction' or 'reductio ad absurdum':

> *The proof is by reductio ad absurdum, and reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.*

The proof he is referring to here is the one given by Euclid to prove the proposition that there are infinitely many prime numbers. It is a proposition for which new proofs continue to be found even in the present day. The same can be said about the Pythagorean theorem (and in this issue itself, we present a new and very novel proof found by two teenagers). For anyone who does not have a feeling for mathematics, this drive to find new proofs of a proposition that has already been proved dozens of times over would seem perfectly baffling! But this drive lies at the heart of mathematics.

As mathematics teachers, can we convey this feeling to students? It is important that we talk about it and attempt to show that proof in mathematics is worth studying and appreciating. We may fall short, because the obstacles are many, but it is important that we try.

What are the obstacles? Proof-writing requires a slowing down of our thought processes; one must go slowly and deliberately, making sure that there are no gaps in our reasoning; making sure that we do not miss out any possibilities. Thinking in this manner does not come naturally to us; we tend to think intuitively and inductively, generalising from specific instances (but not realising that we are doing so). It therefore becomes the task of the mathematics teacher to show that intuitive leaps and inductive thinking can lead us astray. But it is also the task of the same teacher to show the great value in problem-solving of intuitive leaps and inductive thinking! This only goes to demonstrate the great complexity of our task.

If we are to take this challenge seriously, then we must begin early and we must begin small. We must introduce students to proof-writing at the primary level, using appropriate contexts. Whole number arithmetic offers many possibilities in this regard. For example: *Why is the sum of two consecutive whole numbers always odd? Or: Why is the sum of three consecutive whole numbers always a multiple of 3?* Or: *Why is the sum of four consecutive whole numbers never a multiple of 4?* (The *Pull-out* describes more examples of this kind.) There are also puzzles and number games that can be used to devise appealing problems; e.g., cryptarithms; coin-weighing problems. Modular thinking is of great value here: dividing the problem into smaller chunks, and tackling each sub-problem completely. We see here the close relationship between problem-solving and proof-writing.

We may also regard proof-writing as an aspect of *communication*: i.e., communicating one's reasoning and thought processes precisely, accurately, and without ambiguity. This reveals yet another facet of proof-writing: its close relationship with language and therefore with thinking itself.

It will be good for mathematics teachers to come together and work out ways to make the writing of proofs an essential and enjoyable part of the teaching-learning of mathematics.

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