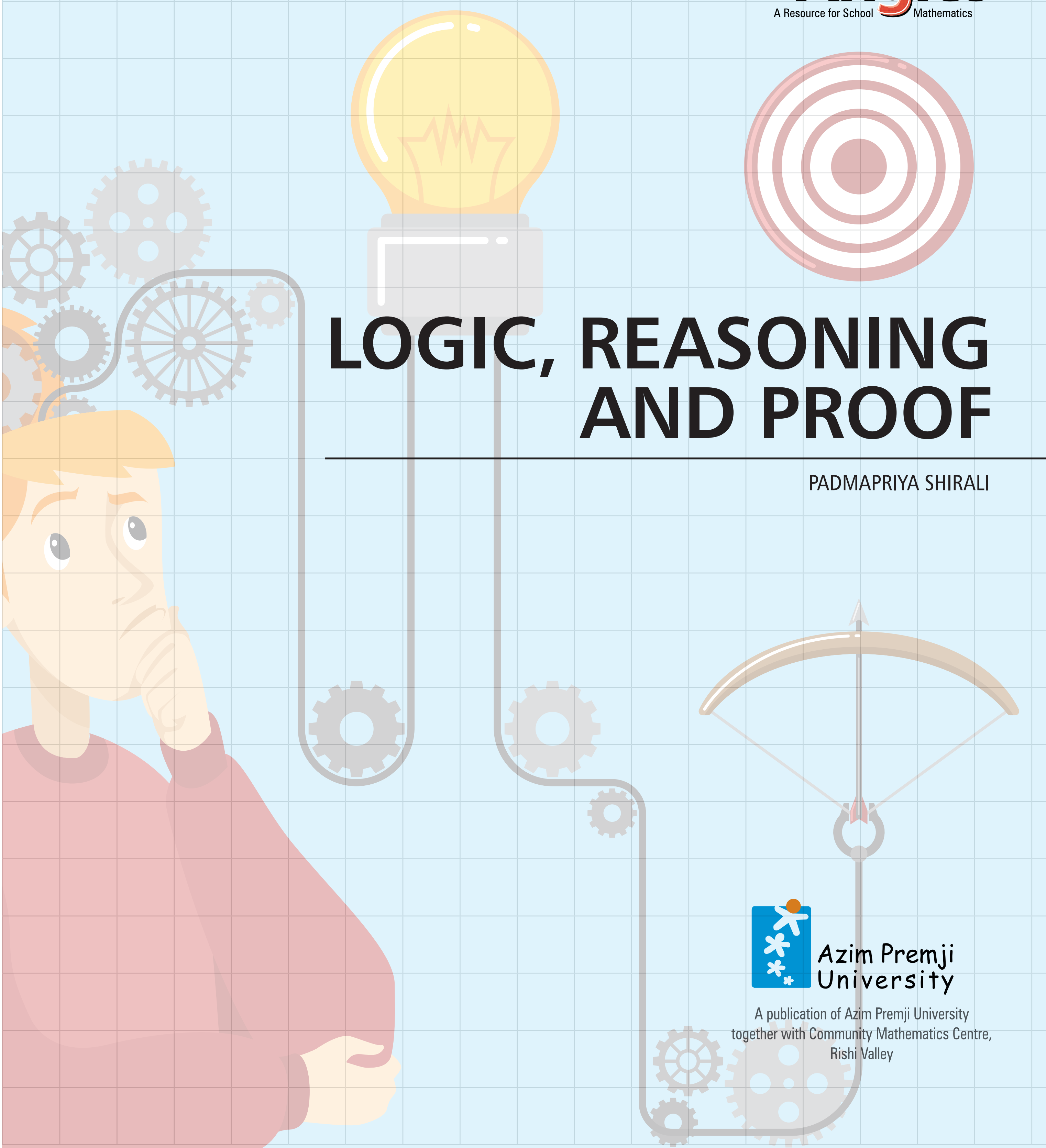
At Right Angles
A Resource for School Mathematics



LOGIC, REASONING AND PROOF

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LOGIC, REASONING AND PROOF

The first steps to proof, which is formally introduced in high school, are logic and reasoning. The school curriculum and the problem selection right from primary grades needs to bring about this aspect in a conscious manner. The curriculum should provide for questions that require justification.

At the primary level logical thinking happens in relation to concrete events and experiences. Children make logical connections through experience, knowledge, interactions, and form conclusions. There is a process of trial and error through which they internalise, arrive at conceptual understanding and logical thinking begins to take shape. As children get older, 10 years and above, they begin to think logically about abstract concepts.

Problem solving, in general, does make use of logic and it is difficult to classify problems under the category of logic. However, some problems can be identified as those that require computational skills, procedural knowledge and that are dependent on memory and conceptual clarity. Some problems require application of logic and an ability to see connections and bring forth a deeper understanding of the properties or relationships that exist between numbers or shapes. They enhance the ability of the students to think through problems and apply strategies for solving them. They build the ability to make a conjecture or an educated guess and to prove it. Students also begin to appreciate how to show proofs without words.

Exposure to problems of this nature will slowly build the skills of reasoning, justification and leads to clarification of concepts. Problems of this nature can be worked on either singly or in pairs to be followed up by discussion/ presentation to the rest of the class.

Naturally, in this process, the first stage for students is to feel completely convinced by the logical thought process that they have used. The second stage is to be able to communicate their thought processes to another student. The third stage is to be able to either satisfactorily answer any questions that are raised or a challenge posed to the thinking process that has been used.

In a classroom, giving scope for raising and inviting questions, encouraging students to make predictions, explore and observe patterns, and make connections will enhance logical thinking. As we give more and more opportunities for children to articulate their reasoning, the level of the engagement with which students approach problems increases. There is a greater likelihood of developing a sense of confidence and greater clarity.

There is good scope for providing such questions in all areas of mathematics and at all levels. The complexity of such problems can be increased gradually. In the initial stages, the problems may require a single step reasoning, later two step reasoning and so on. Students may also employ different modes of representation to prove their point. Some may use drawings, some writing, some symbols. Through the process of sharing, they learn to refine their thinking and presentation methods, and notice the flaws in their arguments. It helps students to develop a greater level of skepticism.

An investigation leading to a discovery followed by a proof can be highly satisfying for any budding mathematician.

Keywords: Proof, questioning, logic, reasoning, justification, conjecture, number, shape, pattern

"To Prove something is to Demonstrate its Truth."

Students should have exposure to informal ways of presenting their proofs in many contexts before being exposed to formal proof writing. Proof requires sequential reasoning which is to be built gradually. Logical thought process must be firmly established before being required to write and aim for rigour in writing. Understanding and proof evolve together.

The process of proof is a movement from selection of some properties of a given fact (premise) to arrive at a conclusion. The given information may have multiple facets and depending upon the facet selected one may arrive at a particular conclusion.

What is a proof? It is a logical argument that establishes the truth of a statement. What is logic in turn? Is it a series of steps where each is derived from the earlier steps? The process involves deduction. In real life we use logical thinking at various points of time to make decisions. While facing a problem we use the available facts to solve the problem.

For something to be called a proof, is it adequate if a statement works for a particular situation? Is it adequate if it works for many situations? Also, to verify the truth of something students need to be exposed to statements which are true under certain given conditions and learn to qualify their statements.

How do children prove something? They may give examples as proof. While an example can work as a demonstration of a principle, it does not suffice as a proof. Here is an opportunity for the teacher to show that an example-based justification is not adequate as a proof. Students may show something through the form of drawing which is a component of mathematical thinking. Experimentation precedes the reasoning process and often one starts with that step before moving into theoretical explanation.

It is necessary to restate the explanation given by the students in the correct form. It is important also to note the information that is assumed as true.

Time does often become a constraint to persist in the process due to the demands of a heavy curriculum but the time spent on building reasoning brings about a deepening of mathematical thinking.

Research has shown that students are able to appreciate proof even if they are unable to produce it themselves. The implication of this is that exposure to arguments that are meaningful can help them to slowly build skills of reasoning.

Here are some suggestions for introducing proof into the classroom at various levels. Some involve numbers, some geometry, while others involve combinations and graphs.

As one can see, the process of asking why and learning to justify can be developed right from an early stage.

Figure 1

PROBLEM 1

If $X + Y = X + Z$, then can you show that $Y = Z$?

When I posed this question in Grade 4, many students used numbers and drawings to show me why it is true. A few tried to give different values to Y and Z as they were different letters and found that the sum on the two sides were not equal. This required pointers from my side so that they could understand that equality is a given condition for them to work with and see its implication on Y and Z. One student came up with a unique solution by bringing a book and a pencil. He placed them on one side and placed another book and a similar pencil on the other side and said if Y is a pencil, Z will need to be a pencil for them to be equal.

It led to an interesting discussion as some students said that the other pencil was a little shorter and hence, they were not equal.

We ended up talking about when do we say things are equal, when do we say things are the same, and when do we say things are similar.

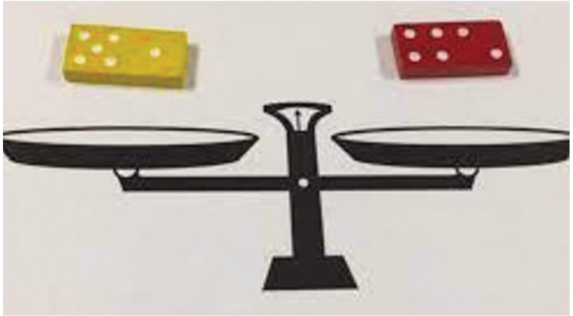

Figure 2

PROBLEM 2

Prove that the sum of two even numbers is always an even number.

Most children attempted this problem by taking two even numbers and demonstrating by example that their sum is even. I did not object to this but prodded them to do it through a drawing. I had introduced even and odd numbers in the earlier year through a pairing activity.

As they made dot drawings to represent the numbers, a few who could recollect the previous process started to circle the pairs in the drawings. Suddenly someone noticed that in each set there was no dot left unpaired and hence both the sets when brought together had to be even. The student was able to convince the others of the logic as it was presented pictorially.

In no time this discovery led to extensions of the same logic to prove two other results.

- Prove that the sum of two odd numbers is always an even number.
- Prove that the sum of one odd and one even number is always an odd number.

Once the idea took root most students used drawings and the idea of pairing to prove the results.

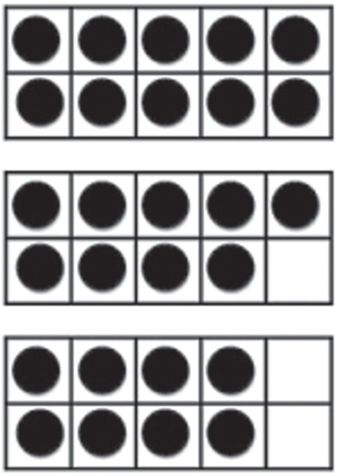

Figure 3

Students began to ask what would happen if we added three odd or even numbers. Would the answer be odd or even?

It was a good demonstration to me on how a few students' learning can affect others and the group can move forward together!

PROBLEM 3: EXPLORATION OF CONSECUTIVE NUMBERS IN GRADE 5

Prove that the sum of three consecutive numbers is a multiple of 3.

1,2,3	2,3,4	3,4,5	4,5,6	5,6,7	6,7,8

We started this problem as an exploration of consecutive numbers and not with the statement given above. Students listed sets of three consecutive numbers. What can we do with these numbers? What happens if we add them? What will happen if we multiply them?

We first summed the numbers and listed the results.

6	9	12	15	18	21

Students noticed that the sums were all multiples of 3. They this with for a few more numbers and saw that the pattern continued.

Why does it happen that the sums are all multiples of 3?

When a question arises out of a discovery, it acquires greater interest as it is not a problem posed by someone else for the student to resolve. They feel an ownership over the problem.

A few attempts were made to explain but were not very satisfactory. I needed to interject and asked them about what they noticed about each set. Various responses came up, and we took note of the fact that each set has one number which is a multiple of 3. In fact, that itself got posed as a question. If you have a set of three consecutive numbers, will it always have a multiple of 3? Why?

Then I asked them what is the relationship of the numbers to one another. Again, various statements came up. The middle number is one more than the one to its left and one less than the one to its right. Another student expressed it as the one on the left is one less and two less than the other numbers. Somebody else said the one on the right is two more than the left most number and one more than the middle number.

One student said that one was a multiple of 3 and the other two together were 3 more than the first.

Figure 4

This statement caused some confusion as the way the student expressed his understanding was not correct. He was referring to the difference of the numbers summing up to 3. The explanation needed to be clarified.

I asked them if they could write each number as a multiple of 3 plus the extra.

We rewrote the sets 3, 4, 5 and 4, 5, 6 and 8, 9, 10 as

$3 \times 1, (3 \times 1) + 1, (3 \times 1) + 2$

$(3 \times 1) + 1, (3 \times 1) + 2, (3 \times 2)$

$(3 \times 2) + 2, 3 \times 3, (3 \times 3) + 1$

Now one student remarked that one number is a multiple of 3, another number is a multiple of 3 and 1 extra, and the remaining number is a multiple of 3 and 2 extra. So, the extras add up to a multiple of 3.

We also drew dot pictures to show the same results.

While students may manage to reasonably justify a result, it is necessary for the teacher to reframe it in precise language.

Very often each proof leads to another result to be proved.

The next challenge was to prove that the sum of three consecutive numbers is thrice the middle number.

There was a lot to discover in sets of consecutive triads!

PROBLEM 4

Prove that the sum of two consecutive odd numbers is always a multiple of 4.

This was also attempted in Grade 5. We first listed some consecutive odd number pairs.

1, 3, 5,7, 9
15,17,19, 21
29,31,33, 35

Figure 5

1 + 3	3 + 5	5 + 7	7 + 9	9 + 11	11 + 13

Initially they tested various pairs to check if it were so. Having satisfied themselves that the statement was true for several examples, the challenge was to figure out the reasoning behind it.

I encouraged them to start with sets which had numbers bigger than 4 and depict the number pair with sets of dot drawings. With the experience that the students had gained with earlier problems, they began to look at the numbers as a multiple of 4 and an extra. That helped them notice the relationships between the numbers.

They first wrote 5, 7 and 7, 9 and 9, 11 as 4 + 1, 4 + 3 and 4 + 3, 4 + 4 +1 and 4 + 4 + 1, 4 + 4 + 3, and then:

$4 \times 1 + 1, 4 \times 1 + 3$

$4 \times 1 + 3, 4 \times 2 + 1$

$4 \times 2 + 1, 4 \times 2 + 3$

Eventually they verbalised their understanding as 'each number is a multiple of 4 and some extra'. The extras in the pair add up to 4. So, the sum of two consecutive odd numbers is a multiple of 4.

Later we split the sets 1, 3 and 3, 5 as follows:

$4 \times 0 + 1, 4 \times 0 + 3$	$4 \times 0 + 3, 4 \times 1 + 1$

The students could see that the situation was the same here.

PROBLEM 5

Building exposure to proof and reasoning need not always be through a big result. It can be done by posing statements which are true and false so that students learn to identify non-true statements and begin to give justification for why something is true or false.

Are these True or False? Can you give reasons why they are True or False?

The word 'numbers' here refers to natural numbers.

- When you multiply a number by an odd number, the answer is always odd.
- When you multiply a number by an even number, the answer is always even.
- Doubling a number results in an even number.
- The sum of four even numbers is a multiple of four.
- When you multiply a number by itself, the answer is even.
- Adding three consecutive numbers results in an even number.

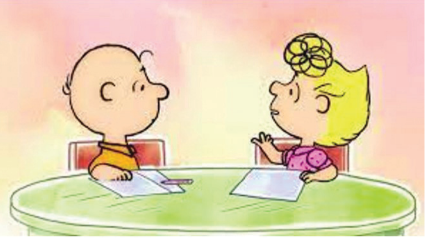

Figure 6

PROBLEM 6

Students should also learn to see when something is always true, when it is true for certain situations and when something is totally untrue. This can help in paving the way for the future when they come across if and only if situations, conditional statements in proofs.

The teacher can also give examples of some statements that are sometimes true. Ex. When you add two numbers you get the same result as when you multiply them. This statement is true when both numbers are zero, or both numbers are 2. But it is not true when the numbers are 2 and 3, for instance.

Are these statements always true, sometimes true, or never true?

- If a number is a multiple of 10, it is also a multiple of 5.
- If a number is a multiple of 4, it is also a multiple of 8.
- If a number is a multiple of 9, it is also a multiple of 2.
- Adding two consecutive multiples of 5 will give a multiple of 10.
- Adding 5 consecutive multiples of 2 will give a multiple of 10.

It is also necessary to look at vice versa.

- A square is a special rectangle. Is a rectangle a special square?

Figure 7

PROBLEM 7

Proof problems can also be posed from geometry and shapes. Here is a problem I tried with Grade 6.

What is the minimum number of faces that a 3D-shape with all plane faces (a polyhedron) can have?

When I posed this question to the students, most of them thought of a cube or cuboid shape, and confidently declared that 3D shapes will have a minimum of 6 faces. I asked if it were so.

More probing brought out other 3D-shapes that have fewer faces than 6, like pyramids and tetrahedral packs ('tetrapaks').

The question now was how can you show that all 3D-shapes have at least four faces?

This was the first time that students were encountering the usage of the phrase 'at least' and it needed some explanation.

We had to do a practical study of various shapes before we could think about the question without reference to a shape. I think it is important to let students refer to their concrete experience before they can abstract out the essentials to visualise it on paper.

Figure 8

We then looked at the minimum number of points that were needed to make a 2D-shape. That was simple as students were aware of triangles; they said 'three.' The question then was what is needed to create the third dimension? Would one point suffice?

If we make a triangle and use the minimum number of points to create a 3D-shape, how many faces would such a shape have?

Soon everyone could see the justification for the statement that 'a 3D-shape will have a minimum number of 4 faces.'

PROBLEM 8

"When you cut off a piece of a shape, you reduce both its area and its perimeter."
Is this always true, sometimes true, or never true? Is it true only under certain conditions?

Again, students can experiment with cutting paper shapes if necessary. Our intention is to focus on arriving at understanding through logic eventually but experimentation may precede the reasoning process.

Figure 9

When I posed this question to students in Grade 6, they were initially quite certain that both the area and perimeter would reduce. Their logic for area was that if a shape occupied a certain space earlier and if the shape was cut and hence made smaller it should occupy less space. This made sense.

But they also argued that a smaller shape should have a smaller perimeter. I asked if it was so?

That began to create some doubt in them and made them wonder about the truth of their statement.

We took out some paper shapes and cut them to test their perimeter lengths. We tried various kinds of cuts, zigzag cuts, curved cuts, and stepped cuts.

Slowly we moved on to the question of an exploration of what happens to a straight line when it begins to get replaced with a zigzag line or a wavy line. We also ended up with a discussion on the shortest distance between two points.

It became evident that cutting off a piece can affect the perimeter in three different ways. The perimeter can stay the same (fig 10b), perimeter can reduce (fig 10a) and perimeter can also increase (fig 10c) depending upon the nature of the cut.

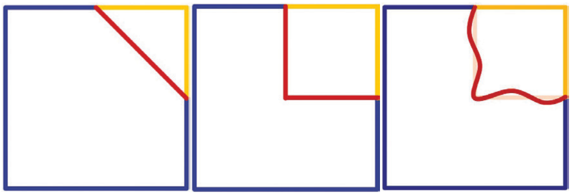

Fig 10a Fig 10b Fig 10c

This was a demonstration to me on how a simple question can lead to a lot of exploration.

What happens if the paper is folded as a rectangle and an L-cut is made in the middle on the fold?

PROBLEM 9

Here is a problem which can be tried with students of Grade 6 or 7.

Is the statement "the value of double the number is always greater than the number" true? Justify your answer.

At the outset the statement may seem true to many students, but once they begin to look at different types of numbers, they will see where it does not always hold true.

Figure 11

PROBLEM 10

"If a number is divisible by 10 and is also divisible by 15, then it is divisible by 150." Is this statement true? Justify your answer.

While solving proof problems students need to bring to the fore their knowledge of concepts and facts that have been learnt earlier. In the given problem students will use their knowledge of factors, prime factorisation and lowest common multiple in reasoning out the falsity of the statement.

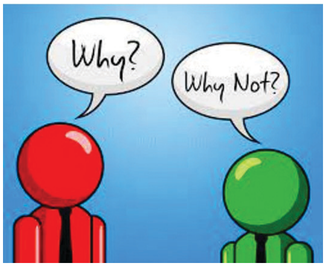

Figure 12

PROBLEM 11

Prove that the product of two consecutive whole numbers is always even.

Students will make use of their knowledge of multiplication as repeated addition in providing their reasoning for this statement.

PROBLEM 12

Here is a problem which can be tried at Grade 6, 7 level.

Prove that when b is a positive integer, the value of $3b$ is always a factor of the value of $12b$.

Students will use their knowledge of positive numbers, factor-multiple relationships to establish the truth of this statement.

A further exploration can occur whether this statement holds for negative integers.

The teacher may like to extend it further and introduce the idea of generalisation by replacing 3 by k and 12 by nk.

Figure 13

PROBLEM 13

By posing problems which involve a good understanding of number properties and laws of arithmetic students are challenged to delve deeper into their grasp of the subject matter.

Prove that the difference of two multiples of 3 is itself a multiple of 3.

What facts and laws of arithmetic will students apply in order to reason out their answer to this poser?

1	2	**3**	4	5	**6**	7	8	**9**	10
11	**12**	13	14	**15**	16	17	**18**	19	20
21	22	23	**24**	25	26	**27**	28	29	**30**
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Figure 14

PROBLEM 14

Here is a problem which I have tried at Grade 7, 8 level.

Prove that when one is added to the product of two consecutive positive even numbers, the result is a square number.

I have found it useful to urge students to use dot arrays in trying to find a proof to this problem. As they juggle around with the dot array and rearrange the dots, they begin to perceive how the rearrangement leaves behind one missing dot to form a square array.

Students who are comfortable with algebraic manipulation may find an algebraic proof for this problem.

The teacher can link the dot arrangement and the algebraic proof.

PROBLEM 15

Problem involving permutations and combinations:

How many different 3-digit numbers can you make using 1, 2 and 3? Can you justify that as the maximum number?

These problems make use of systematic ordering strategies as a way of approaching solutions. Students begin with one number and work out the possible combinations before moving onto the next.

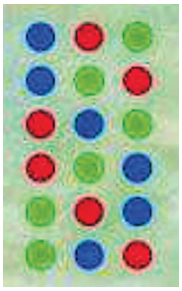

Figure 15

PROBLEM 16

Encourage students to become comfortable with algebraic expressions and defend their understanding.

Which is bigger, $3n$ or $n + 3$? How will you justify your answer?

PROBLEM 17

Here is a problem which can be tried at Grade 8 level.

Three numbers m, n, p have the property that m divides n, n divides p, and p divides m. What must be true about these numbers? Prove your conjecture.

Figure 16

PROBLEM 18

Another problem at Grade 8 level.

Given two distinct numbers, X and Y, prove that their mean $(X + Y)/2$ lies between X and Y.

PROBLEM 19

A problem which requires a good understanding of the multiplication process and associated logic.

What is the largest digit in the product 11111111 × 11111111? How can you be sure you have the right answer?

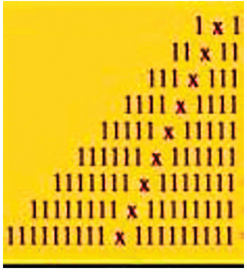

Figure 17

PROBLEM 20

Another problem involving permutations. Permutations and combinations are a topic in which problems involving reasoning and supporting arguments are easy to find.

How many three-digit numbers containing only even digits are divisible by 9? Can you justify your answer?

Figure 18

PROBLEM 21

Here is one for Grade 7 involving the concept of rounding.

When a certain four-digit number is rounded to two significant figures, the answer is 8000. What is the greatest value the number could be? What is the smallest value the number could be?

Justify your answer.

PROBLEM 22

Here are a few problems for middle school students for group work.

Credit: https://www.youcubed.org/tasks/paper-folding/

Start with a square sheet of paper and make folds to construct a new shape. Explain how you know the shape you constructed has the specified area.

- Construct a triangle with exactly 1/4 the area of the original square. Convince your partner that it has 1/4 of the area.
- Construct another triangle, also with 1/4 the area, that is not congruent to the first one you constructed. Convince your partner that it has 1/4 of the area.
- Construct a square with exactly 1/2 the area of the original square. Convince your partner that it is a square and has 1/2 of the area.
- Construct another square, also with 1/2 the area, that is oriented differently from the one you constructed in task 3. Convince your partner that it has 1/2 of the area.

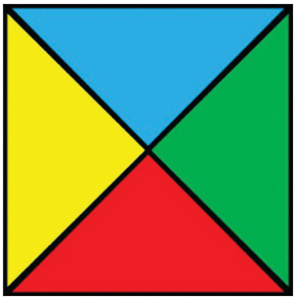

Figure 19

PROBLEM 23

Credit: NRICH

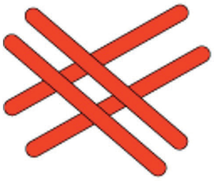

Figure 20

Figure 21

There are four sticks (two sets of parallel sticks) which make four crossings.

How many crossings do five sticks (with two sets of parallel sticks) make?

Still keeping two sets of parallel sticks, this time with seven sticks in total, can you arrange them in another way, to get a different number of crossings?

- What is the least number of crossings you can make?
- What is the greatest number of crossings you can make?
- Can you find all possible numbers of crossings with seven sticks?

What do you need to do to prove that you have them all or how could you show that you have them all?

PROBLEM 24

Here is a problem for Grade 8.

In problems that are posed it is important to select the appropriate ones and to be clear of the ideas that one is trying to communicate and whether students can attempt the questions. If they are too difficult and students cannot make any headway into them, then the purpose is lost.

The lengths of the sides of a right-angled triangle are all integers. Prove that if the lengths of the two shortest sides are even, then the length of the third side must also be even.

Most students use algebra and their knowledge of Pythagoras theorem to solve this problem. Is there a geometric solution to this problem?

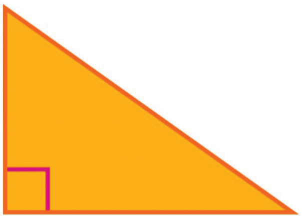

Figure 22

PROBLEM 25

Here is a question based on a proper understanding of distance-time graphs.

"If a person walked around in a circle around his home, the time-distance graph would be like a circle." Is this statement true or false? Justify your answer.

Figure 23

PROBLEM 26

Show that if you add 1 to the product of four consecutive numbers, the answer is always a perfect square.

Figure 24

We experimented with some sets of four consecutive numbers.

1,2,3,4	2,3,4,5	6,7,8,9	11,12,13,14

The product of 1,2,3,4 is 24, and 24 + 1 = 5^2.

The product of 2,3,4,5 is 120, and 120 + 1 = 121 = 11^2.

The product of 6,7,8,9 is 3024, and 3024 + 1 = 3025 = 55^2.

The challenge was to find the proof. We tried the algebraic approach and found ourselves stuck with complicated expressions like $a^4 + 6a^3 + 11a^2 + 6a + 1$ which was not helpful.

We looked again at the numbers arising in the problem and noticed a very distinctive pattern when we multiplied the two numbers at the extremes and the two numbers in the middle:

- In the case of 1,2,3,4, we noticed that 1 × 4 = 4, 2 × 3 = 6, and 5 is between 4 and 6.
- In the case of 2,3,4,5, we noticed that 2 × 5 = 10, 3 × 4 = 12, and 11 is between 10 and 12.
- In the case of 6,7,8,9, we noticed that 6 × 9 = 54, 7 × 8 = 56, and 55 is between 54 and 56.

In each case, the two products obtained were seen to be consecutive even numbers.

We now experimented with dot drawings and rearrangements to slowly unravel the proof. Here is how it worked out in the case of 1,2,3,4. We first constructed the dot diagram for 4 × 6:

o o o o o o
o o o o o o
o o o o o o
o o o o o o

Removing the last column, converting it into a row, and then placing it at the bottom of the diagram, we obtained the following:

o o o o o
o o o o o
o o o o o
o o o o o
o o o o

We immediately saw that we have a square array with exactly one missing unit. We now understood why $(4 \times 6) + 1$ is a square: $(4 \times 6) + 1 = 5^2$.

We then worked with another set 2,3,4,5 and tried the same approach: $2 \times 5 = 10$, and $3 \times 4 = 12$.

Here are the pictures for the two cases, using a grid rather than a dot array:

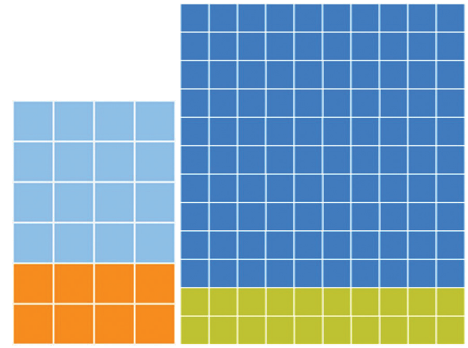

Figure 25

(Credit for the two pictures: Swati Sircar)

We realised that this process will work for the product of any four consecutive numbers.

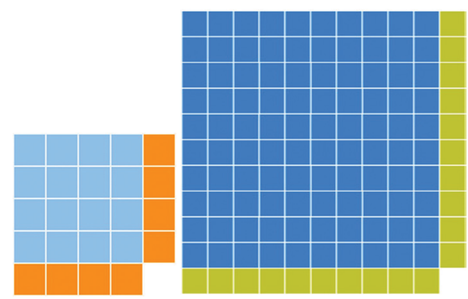

Figure 26

PROBLEM 27

Here is a problem for Grade 8.

Here are two visual aids to help students to prove the given results. Students need to articulate their interpretation of the drawings. Teachers may need to ask some leading questions to start the students in reading from the diagram. They will then need to use their knowledge of area and algebra to arrive at the proof.

How does this diagram show that the sum of a positive number and its reciprocal is at least 2?

Credit: (Nelson, p. 62)

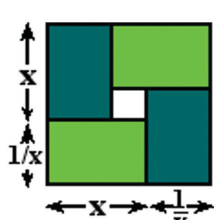

Figure 27

How do the two pictures below "prove" that $a^2 + b^2 \geq 2ab$?

Students will need to notice the difference between the two pictures to interpret the way the reconstruction has happened and use their knowledge of area and deductive reasoning to arrive at the proof.

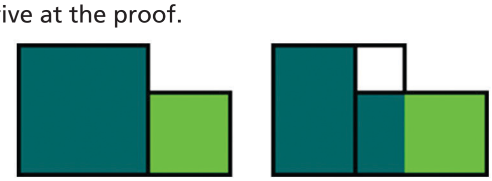

Figure 28

PADMAPRIYA SHIRALI

PADMAPRIYA SHIRALI is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. In the 1990s, she worked closely with the late Shri P K Srinivasan. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box.' Padmapriya may be contacted at padmapriya.shirali@gmail.com.