

Exploring Geometrical Constructions – The GeoGebra Way

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Performing constructions of geometrical figures using a compass and straight edge (that is, a ruler without markings) are an integral part of learning mathematics at school. These are typically introduced in the middle school and students generally enjoy this topic as it gives them an opportunity to work with their hands. Teachers spend a significant amount of time teaching the “steps of construction” which students follow as an algorithm to arrive at the correct constructed figure. However, this approach does not leave much scope for exploration of figures and their geometrical properties. In this article, we shall illustrate the opportunities provided by GeoGebra, a Dynamic Geometry Software (DGS), in exploring constructions of simple geometrical figures through multiple approaches.

The reader may wish to read the introductory article *Dynamic Geometry Software: A Conjecture Making Tool* which appeared in the Techspace section of the July 2020 issue, in order to recall some basic features of GeoGebra.

Constructing an equilateral triangle

An equilateral triangle is one of the easiest figures to construct. Usually one of two methods, using compass and ruler, are taught to students.

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Method 1: Draw a straight line segment AB assumed to be the base of the required triangle. Place the compass point on vertex A, stretch the pencil point of the compass to vertex B and draw an arc somewhere above the line segment AB. Now place the compass point on vertex B and using the same length draw a new arc, which cuts the previous arc. The point of intersection of these two arcs is the third vertex (say C) of the equilateral triangle. Once vertex C is joined to A and B we have an equilateral triangle. See Figure 1(a).

Method 2: Draw a straight line segment AB, which is assumed to be the base of the required triangle. Construct an angle of 60° at vertex A and another one at vertex B. Produce the arms of the angles at A and B till they meet and we have an equilateral triangle ABC. See Figure 1(b).

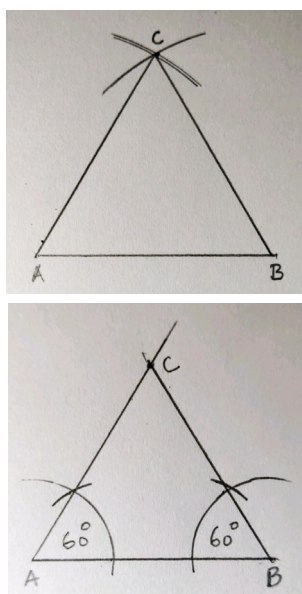


Figure 1. Compass and ruler methods of constructing an equilateral triangle: (a) using the property of equality of sides, (b) using the property of equality of angles

Although both methods produce equilateral triangles, students are often unable to provide a reasoning or justification as to why these methods work. Method 1 uses the property that all sides of an equilateral triangle are equal while method 2 is based on the fact that all angles have a measure of 60° each. However, these properties may not be visually evident while following the “steps of construction” using a compass and ruler.

A group of grade 6 students were given the opportunity to explore the construction of geometrical figures using GeoGebra. As they did not have any prior exposure to using GeoGebra, they underwent a 90-minute session in which they were familiarized with the basic construction tools and were also given time to explore these tools. Following this, they were required to construct two figures using GeoGebra – an equilateral triangle and a regular hexagon.

While constructing an equilateral triangle some interesting initial responses were elicited by students. Some used the **Polygon** tool to draw a triangle (Figure 2(a)) and then tried to make it an equilateral triangle through manual adjustment, that is, by dragging one of the vertices and adjusting the side lengths. It was pointed out by their teacher that if by dragging a vertex, the triangle loses its properties, that is equality of sides and all angles being 60° , then the construction is not a robust one. At this point students were introduced to the idea of a “drag test” in GeoGebra. The drag test is a dragging strategy that can facilitate generalization since its purpose is to let a DGS user discern or verify invariants under varying appearances of the object of exploration. A robust or correctly constructed figure, which does not get “messed up” upon dragging is said to survive the “drag test”. Another group of students used the **Regular polygon** tool to construct an equilateral triangle. This indeed led to a robust equilateral triangle, which did not lose its properties upon dragging (Figure 2(b)). The Algebra view was used to verify that all sides were equal and all angles measure 60° . However, a few other students felt that this was “cheating” as the **Regular polygon** tool was an inbuilt feature of GeoGebra and therefore the figure that emerged was not actually a constructed one! There was some difference of opinion among students on this matter. Taking advantage of the situation, the teacher then asked the students to construct an equilateral triangle using GeoGebra’s construction tools. She prompted students to use the **Segment** tool, **Circle** tool, and **Angle** tool for completing the construction.

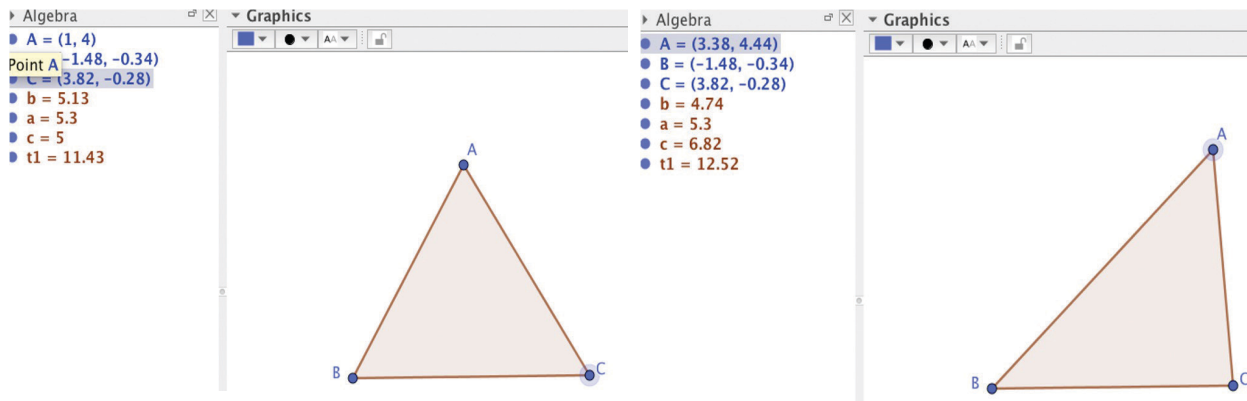


Figure 2(a) Students used the **Polygon** tool to draw an equilateral triangle via manual adjustment. However, on dragging a vertex the triangle was no longer equilateral.

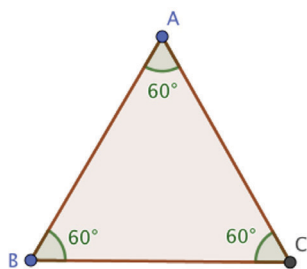


Figure 2(b) The Regular polygon tool led to an equilateral triangle, which retained its properties upon dragging.

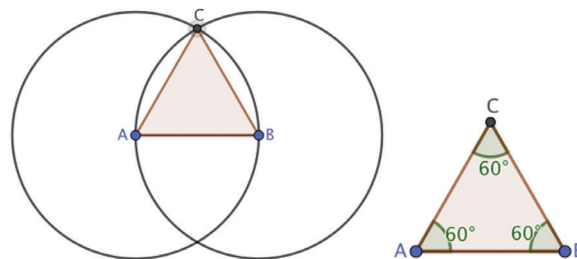


Figure 3(a) Students used the **Segment** tool, **Circle** tool and **Polygon** tool to construct an equilateral triangle. The hide option was used to remove the circles and the **Angle** tool was used to verify that all angles measure 60° .

After this the students at once set to work in pairs. Most of their constructions may be classified under one of the following two approaches.

Approach 1: Students drew a line segment AB using the **Segment** tool. Using A as centre and AB as radius, a circle was drawn using the **Circle with centre and point** tool (available within the **Circle** tool). Similarly, using B as centre and AB as radius, another circle was drawn which intersected the previous circle. One of the intersection points of the two circles was selected as the third vertex of the triangle. The **Polygon** tool was then used to complete the triangle (See Figure 3(a)). When asked for the rationale for this approach, some students expressed that they were trying to imitate what they would do using a compass and ruler on paper. However, instead of drawing arcs, they were drawing circles. A student argued, “drawing full circles helps to

see that all three sides are equal since these sides are radii of equal circles”. It was heartening to see that students were eagerly providing explanations for the way they chose to proceed with their construction.

Approach 2: Students drew a line segment AB using the **Segment** tool. At A, a 60° angle was constructed using the **Angle with given size** option (available within the **Angle** tool). Similarly, a 60° angle was also constructed at B. However, it took students some time to understand the clockwise and anticlockwise orientations while constructing the angles. Once this was done, they were able to identify the third vertex of the triangle. See Figure 3(b). They also measured the third angle to ensure that it had a measure of 60° .

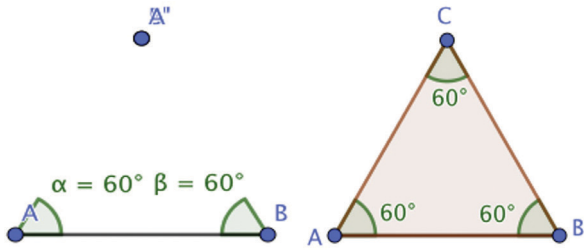


Figure 3(b) The **Angle with given size** option was used to construct angles of 60° at vertices A and B respectively and the **Polygon** tool was used to complete the equilateral triangle.

Both approaches led to equilateral triangles, which survived the drag test. Thus by dragging a vertex of the constructed triangle, they could modify the side lengths and the orientation of the triangle, but its fundamental properties, namely equal sides and equal angles remained invariant in a dragging episode. This led them to experience varying appearances of an equilateral triangle in a visual-dynamic way. While dragging parts of a figure, the **Algebra view** as well as the **Graphics view** revealed those attributes, which vary and those, which remain invariant. Being able to discern what varies and what remains invariant is key to experiencing a mathematical property. According to Leung (2003)

A key feature of DGS is its ability to visually represent geometrical invariants amidst simultaneous variations induced by dragging activities...

...when engaging in mathematical activities or reasoning, one often tries to comprehend abstract concepts by some kind of “mental animation”, that is, mentally visualising variations of conceptual objects in the hope of “seeing” patterns of variation or invariant properties (p. 197)

The next task posed to the students was to construct a regular hexagon (without using the **Regular polygon** tool). Before beginning the task, students were encouraged to list out the properties of a regular hexagon, which would help them in their construction. Thus, all sides are equal, all interior angles measure 120° , exterior angles measure 60° and that

a regular hexagon can be divided into six equilateral triangles were pointed out as some of the properties of a regular hexagon. The teacher asked them to use these properties in conceptualizing their construction process. The end result was to be a hexagon, which would survive the drag test.

The hexagon construction led to many interesting responses. One group of students used the fact that every interior angle measures 120° . They drew a line segment AB and used the **Angle with given size** from the **Angle** tool menu to draw angles of 120° both at A and B. They were familiar with the clockwise and counterclockwise orientation from the equilateral triangle construction earlier. As new vertices of the hexagon emerged, they constructed more interior angles of 120° and finally completed the hexagon. See Figure 4(a).

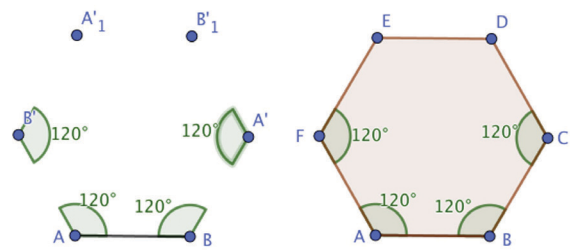


Figure 4(a) The **Angle with given size** option was used to construct a regular hexagon.

Another group wanted to use the fact that a regular hexagon comprises six equilateral triangles. These students began the construction with an equilateral triangle following which they used the **Reflect about Line** option to reflect the triangle about its sides. Repeating the reflection process, they completed the hexagon. See Figure 4(b).

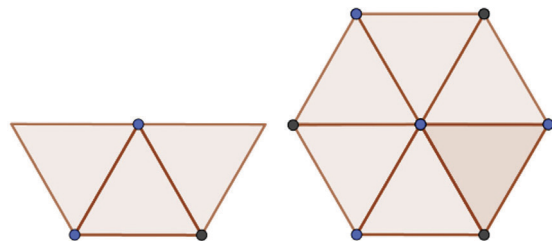


Figure 4(b) The **Reflect about Line** option was used to construct a hexagon. An equilateral triangle was constructed and was reflected about one of its sides.

A third group of students constructed six intersecting circles of the same radii. After this, they selected the six points of intersection of the pairwise intersecting circles to construct a hexagon. Later they used the hide/unhide feature to remove the circles leaving on the screen a regular hexagon. See Figure 4(c).

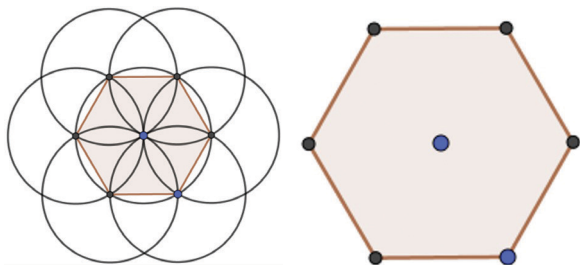


Figure 4(c) Students constructed a hexagon by joining the points of intersection of six intersecting circles of the same radii.

It was interesting to see that students could approach the construction of the hexagon using their own reasoning and their knowledge of GeoGebra tools. Feedback taken at the end of the session revealed that most students considered the use of GeoGebra “very interesting” and “more accurate than compass and ruler constructions”. They also opined that GeoGebra enabled them to construct a figure in multiple ways. Further, they expressed that constructing a regular hexagon was a novel experience for them. A student also commented that the GeoGebra method of constructing the hexagon would also enable her to do the construction using compass and ruler.

Constructing a square

In another study, grade 7 students were asked to construct a square using GeoGebra. Although they had constructed squares and rectangles in their regular classes using compass and ruler, they had no prior experience in using GeoGebra. After going through a 90-minute session on exploring the basic construction tools, their initial response was to draw a square using the **Polygon** tool through manual adjustment (See Figure 5(a)). However, upon dragging a vertex, the figure would lose its properties and it would no longer be a square. The students

then discovered the **Regular polygon** tool. Using this they were able to construct a robust square, which survived the drag test. (Figure 5(b)). Later, when the researchers asked the students to construct a square using GeoGebra’s construction tools, they appeared to use their prior knowledge of constructing a square using a compass and ruler to figure out the construction process in GeoGebra. One of their approaches was to construct perpendiculars at end points A and B of a line segment AB using the **Perpendicular line** tool. Students further recalled that arcs had to be drawn in the compass and ruler method. When asked why the arcs were required, one student said “the arc is needed to mark the point on the perpendicular line so that the sides of the square remain equal to the line segment AB.” After this, they used the **circle tool** to complete the construction of the square (See Figure 5(b)). Any errors in the construction process were rectified using the hide/unhide option. In fact, the GeoGebra construction required more mathematical knowledge about the properties of a square and circle than the compass-ruler construction. Contrasting between the static construction using compass and ruler and the dynamic construction on GeoGebra, a student commented “when we construct a square using compass and ruler, like the way our teacher taught us, we cannot do anything to it afterwards. It cannot be changed unless we use an eraser. But this GeoGebra square can be moved and changed in both position and in size and even after dragging it remains a square!” When asked to check if the constructed figure is actually a square, students resorted to the **Algebra view** and verified that all the sides were equal and all angles were right angles. By dragging, students learnt that the properties of the square which are relevant to the construction process are equality of sides and all angles being right angles (which should remain invariant) whereas the side length of the square or its orientation are irrelevant (and may vary). This illustrates well the “drag for contrast approach” by which students were able to construct a ‘robust’ square, which ‘survived the drag test’. According to Leung (2002):

Contrast is about seeing differences, comparing between what is and what is not, hence anticipating (conjecturing) what can be and cannot be (p. 6).

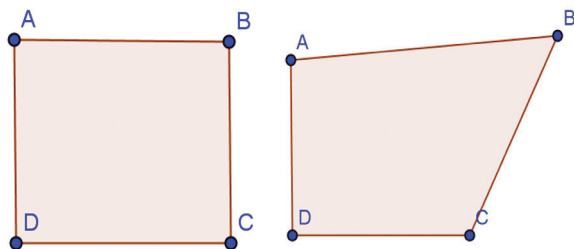


Figure 5(a) A square drawn via manual adjustment does not retain its properties if one of its vertices is dragged.

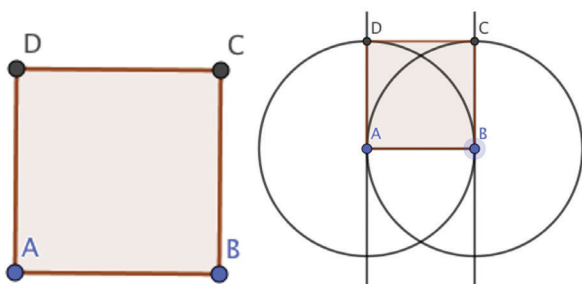


Figure 5(b) Squares constructed using the **Regular Polygon** tool and tools such as **Perpendicular line**, **Circle with Centre and Point** lead to robust squares, which survive the drag test.

Constructing a rectangle

As a natural extension to the square construction activity, the grade 7 students wanted to construct a rectangle. The researcher asked them to construct a rectangle using GeoGebra's construction tools. Students felt that this was an easy task as it would be very similar to constructing a square. They drew a line segment AB using the **Segment** tool and then used the **Perpendicular line** tool to draw perpendiculars to AB at A and B respectively. Once this was done, they realized that the next vertex, C, of the rectangle can be chosen anywhere on one of the perpendiculars. Once C is chosen, a line parallel to AB was drawn through C. The **Intersect** tool was then used to identify the fourth vertex D (see Figure 6(a)). Hiding the perpendicular and parallel lines led to a robust rectangle, which maintained its properties upon dragging. Students observed that in this

“draggable dynamic rectangle” (a term coined by the students), the opposite sides are equal and all angles are right angles. The researcher asked if the rectangle could be dragged into a square. Students dragged one of the vertices A, B or C and observed that the rectangle can indeed be made into a square. See Figure 6(b). However one student remarked that the square they had constructed earlier cannot be dragged into a rectangle. This led to a discussion and students concluded that a rectangle could be made into a square by dragging, since a square is a special kind of a rectangle. However, once a square is constructed it cannot be dragged into a rectangle. It was interesting to see that students' explanations were moving towards the relationship between figures and hierarchical inclusions.

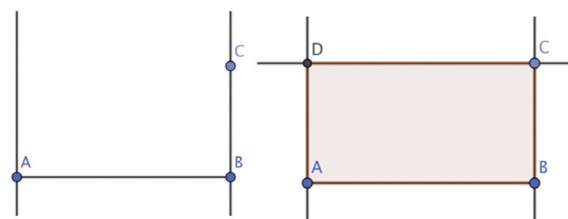


Figure 6(a) A rectangle was constructed using the **Perpendicular Line** and **Parallel Line** tools.

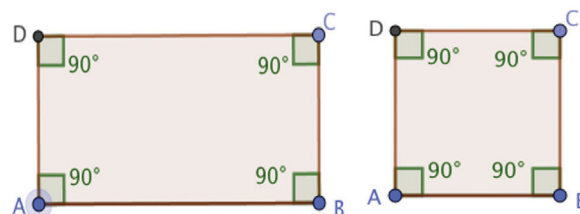


Figure 6(b) Students verified the properties of the rectangle and also dragged its vertices to form a square.

Constructing a rhombus

Asking students to produce constructions, rather than drawings, can help improve their understanding of formal definitions and relationships among geometric objects. A group of grade 8 students who had studied the topic on quadrilaterals were assigned the following task.

Task: Construct a rhombus using GeoGebra and explain why it is a rhombus. Reflect on how this is different from constructing a square.

This group of students already had the prior experience of constructing a square using GeoGebra. They were also familiar with the properties of a rhombus, namely, equality of sides, equality of opposite angles and that diagonals bisect each other at right angles. After recalling these properties students decided that it would be easier to begin the construction of the rhombus with one of its diagonals. A line segment AB was drawn using the **segment tool**. This was followed by constructing its **Perpendicular bisector** (since the other diagonal would be perpendicular to AB). The next step was to select the two vertices (of the rhombus) say C and D, on this perpendicular line, so that the sides AC, BC, AD and BD are equal. Many students selected a point C on the perpendicular line and joined it to the vertices A and B (See Figure 7(a)). A student remarked “As C is on the perpendicular bisector of AB, it will be equidistant from the points A and B”. Following this, they tried to identify a point D on the perpendicular line so that C and D were on opposite sides of AB. Joining D to A and B led to a rhombus like figure. However this would not always remain a rhombus if the point D was dragged on the perpendicular line. (See Figure 7(a)). Students began to explore a way of working around this problem. They realised that the points C and D should be equidistant from the diagonal AB. After some exploration, a student came up with the suggestion “let’s select a point C on the perpendicular line and then use the **Reflect about Line** tool to reflect C about the line AB”. Indeed this led to a point C', the image of C on line AB. This suggestion was appreciated by everyone and it was decided that C' could be renamed as D. After completing the construction, students used the **Angle** tool to measure the angles of the rhombus and also used the **Algebra view** to check if all the sides were equal. This led to a dynamic rhombus, which preserved its properties upon dragging. See figure 7(b).

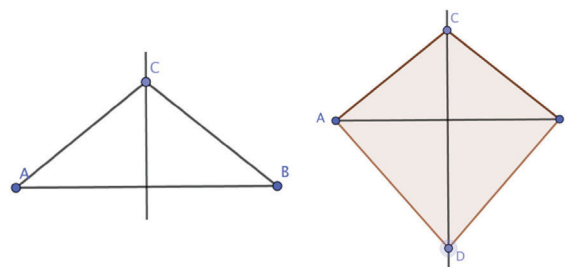


Figure 7 (a) Students’ approach of drawing a rhombus in GeoGebra. By dragging the point D the figure was no longer a rhombus.

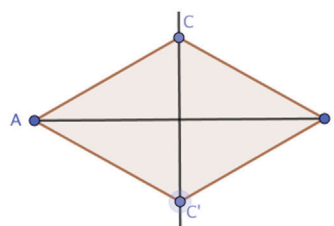


Figure 7(b) Students used the **Reflect about Line** tool to reflect the point C about AB to construct a rhombus.

At this point the teacher decided to introduce students to the use of sliders to construct a rhombus. Two sliders were created. One (named *side*) was used to vary the side length of the rhombus and the other (named α) was used to vary the acute angle (marked in red in Figure 7(c)). By dragging the blob on the sliders, the angles and side lengths of the rhombus could be varied. However, the equality of opposite sides and opposite angles remained invariant. Three examples of rhombuses which emerged from dragging the sliders are shown in Figure 7(c).

This aptly illustrates the *mathematical variability principle* by Zoltan P. Dienes (1963), which states that

as every mathematical concept involves variables, all these mathematical variables need to be varied if the full generality of the mathematical concept is to be achieved. (p. 158)

In Figure 7(c) dragging the sliders can be used to vary the irrelevant properties of the rhombus, namely side length and angle measures.

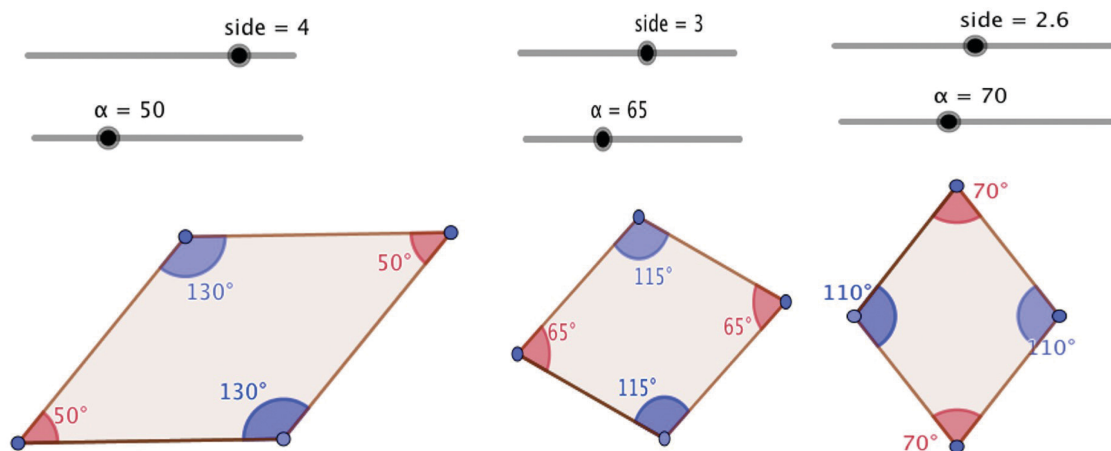


Figure 7(c) Sliders were used to construct a dynamic rhombus in which the length of sides and angle measures could be varied, while the equality of sides and equality of opposite angles remained invariant.

However, in the dragging episode, the equality of sides and equality of opposite angles remain invariant. This contrasting experience of the varying and the invariant enables the user to experience the mathematical concept of the rhombus in its fully general form.

Conclusion

Typically, in the traditional classroom, students are asked to construct geometrical figures using a sequence of steps. This article attempts to provide the reader with alternative ways of constructing the familiar geometric figures. The dynamic nature of the tool allows students to

explore the properties of figures and also use them in the process of construction. From a pedagogical perspective, dynamic geometry software has created new interest in teaching of geometry as it enables students to perform geometric constructions with a high degree of accuracy and also explore constructions in multiple ways using the dragging tool. Experiencing geometrical figures in a visual-dynamic way facilitates the understanding of propositions, making and testing of conjectures, provide explanations and to engage in argumentation finally leading to proof. The reader may wish to explore the construction of other geometrical figures using GeoGebra.

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