Counting Perimeter Magic Triangles

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The problem

Arrange numbers from 1 to 9 in the circles (Figure 1) such that the sum of numbers on each side is the same, without repeating any of the numbers (so each number appears exactly once). Such a configuration is called a perimeter magic triangle of order 4 (or simply a magic triangle; we leave out the word 'perimeter'). The number of circles on each side determines the 'order' of the magic triangle. Is such a configuration possible? If so, how many solutions exist?

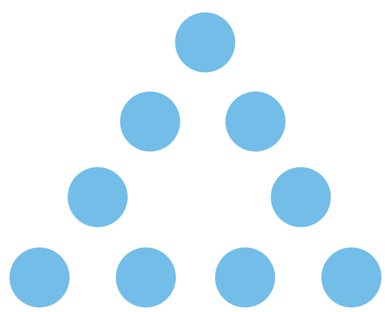

Figure 1. Perimeter magic triangle of order 4

Generalised problem

Construct magic triangles of orders 5, 6, and 7, using the numbers from 1 to 12, 1 to 15, and 1 to 18 (respectively), by increasing the number of circles by 1 on each side in Figure 1. How many solutions are there?

If the number of circles on each side is 4 (as in Figure 1), then the numbers to be used will be from 1 to $3\,(4-1) = 9$. If the number of circles on each side is n, then the numbers to be used will be from 1 to $3(n-1)$.

Keywords: Magic triangle, magic sum, partitions, counting, reasoning

In this article, I study the problem of counting the number of perimeter magic triangles of order n. The numbers used are the following: $1, 2, 3, \ldots, 3(n-1)$. The condition is that the sums of the numbers on the three sides must be equal. It should be clear that manually counting the triangles is very tedious; we require a systematic approach.

Important terms and notation

Below are the definitions of terms that are going to be used to solve the problem.

- **Order:** The number of circles on each side of the Magic Triangle.
- **Magic Sum (S):** The sum of the numbers on every side of the triangle is the Magic Sum of the triangle, denoted by S.
- **Partitions:** A partition of n is an expression of n written as a sum of positive integers. The order in which the integers occur is not important; the sum is an unordered sum. For example, $4 + 2 + 1$ and $3+2 + 2$ are partitions of 7. We are particularly interested in partitions where the summands are *distinct* (i.e., numbers are not repeated). The partitions are then referred to as *distinct partitions*.
- **Middle Numbers:** All the numbers excluding those at the vertices: p, q, r, s, t, u.

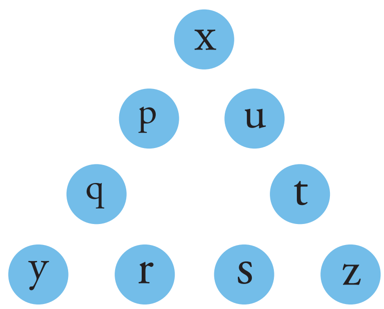

Figure 2

- **Middle Sum:** The sums of all the middle numbers on each side: $p + q, r + s, t + u$.

A method to solve the problem for order 4

Approach 1: (To find the middle numbers in Figure 2)

Step 1: For the magic sum S and vertices (V_1, V_2, V_3), where $V_1 < V_2 < V_3$, the middle sums are $S_1 = S - (V_1 + V_2)$, $S_2 = S - (V_2 + V_3)$ and $S_3 = S - (V_1 + V_3)$.

Step 2: We need to find the number of ways of partitioning (S_1, S_2, S_3) using distinct middle numbers.

Let the partitions be $S_1 = M_1 + M_2$, $S_2 = M_3 + M_4$, and $S_3 = M_5 + M_6$, where the middle numbers and vertex numbers are distinct.

Step 3: Enter each number into the Magic Triangle (Figure 2).

Approach 2: (Finding the bounds on the magic sum)

Step 1: To find the smallest magic sum, the three smallest numbers, i.e., 1, 2 and 3 must be at the vertices. In finding the sum of all the numbers on all the sides, each vertex would be counted twice.

The sum of the numbers on all the sides would be (the sum of the vertices + the sum of all the numbers).

Therefore, the magic sum on each side would be $\frac{\text{the sum of the vertices + the sum of all the numbers}}{3}$.

Step 2:

$$\text{Smallest Magic Sum} = \frac{(\text{sum of the three smallest numbers + the sum of all the numbers})}{3}$$
$$= \frac{(\mathbf{1}+\mathbf{2}+\mathbf{3}+1+2+3+4+5+6+7+8+9)}{3} = 17.$$

Step 3:

$$\text{Largest Magic Sum} = \frac{(\text{sum of the three largest numbers + the sum of all the numbers})}{3}$$
$$= \frac{(7+\mathbf{8}+\mathbf{9}+1+2+3+4+5+6+7+8+9)}{3} = 23.$$

So, for magic triangle of order 4, the magic sum S must satisfy the condition $17 \le S \le 23$.

Similarly, for a Magic Triangle of Order N, the range of the sums on each side is:

$$\frac{1}{3}\left(6+\frac{(3n-3)(3n-2)}{2}\right) \text{ to } \frac{1}{3}\left(9n-12+\frac{(3n-3)(3n-2)}{2}\right),$$
$$\text{i.e., } 2+\frac{(n-1)(3n-2)}{2} \text{ to } 3n-4+\frac{(n-1)(3n-2)}{2}.$$

Now that we have found bounds on the magic sum, we can count the number of magic triangles for each magic sum.

An example of Approach 1

We try to form a Magic Triangle with vertex numbers (1, 2, 3) and magic sum 17.

Step #1: The Middle Sums are $17-(1+2)=14$; $17-(2+3)=12$; and $17-(1+3)=13$.

Step #2: We must find distinct partitions of (12, 13, 14) using the numbers (4, 5, 6, 7, 8, 9).

Partitions of 12	Partitions of 13	Partitions of 14
$8+4$	$9+4$	$9+5$
$7+5$	$8+5$	$8+6$
–	$7+6$	–
Total: 2	Total: 3	Total: 2

Step #3: We form pairs, such that the digits do not repeat as repetition is not allowed.

$$\text{Pairs in Set 1: } (8+4), (7+6), (9+5) \mid \text{Pairs in Set 2: } (7+5), (9+4), (6+8).$$

The resulting magic triangles are shown in Figures 3 and 4.

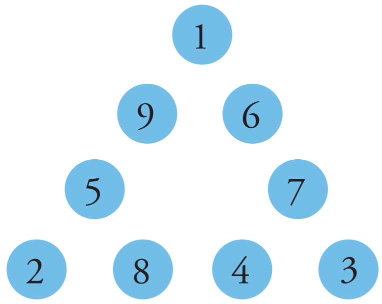

Figure 3. Magic triangle 1

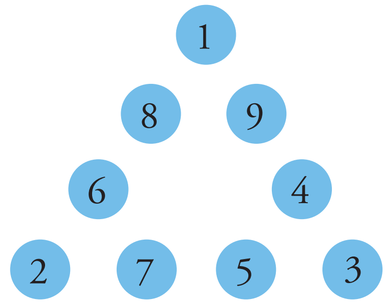

Figure 4. Magic triangle 2

Complement

The lowest number that one can use is 1, while the highest number is 9. Note that $1 + 9 = 10$.

Thus, on taking the complement of 10 from each number, i.e., on subtracting each number in the magic triangle from 10, another magic triangle is formed. In this way, one can generate another magic triangle from an existing magic triangle.

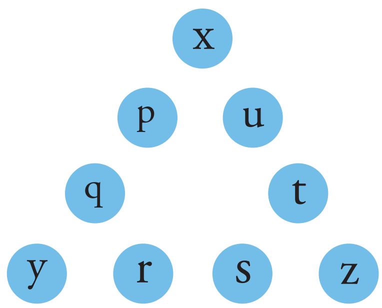

Figure 5. Using complements to generate another magic triangle

In Figure 5, $x + p + q + y = y + r + s + z = z + t + u + x$, so the quantities $(10 - x) + (10 - p) + (10 - q) + (10 - y)$, $(10 - y) + (10 - r) + (10 - s) + (10 - z)$ and $(10 - z) + (10 - t) + (10 - u) + (10 - x)$ are all equal.

If the sum of a magic triangle of order 4 is S, then the magic sum of the new magic triangle would be $40 - S$.

For a magic triangle of order n, the lowest number is 1 and the highest number is $3(n - 1)$. Since $3(n - 1) + 1 = 3n - 2$, on taking the complement of each number from $3n - 2$, another magic triangle is obtained. So, we obtain the same number of magic triangles with magic sum S and $3n - 2 - S$.

The algorithm (for order 4)

Program(In Java) + Output: Generating Magic Triangles - Order 4 (Code + Output)

The first part of the program generates partitions of the middle sum, with a given sum and given vertex numbers. A loop runs which determines the range of the possible sum and the vertices. This is then stored in a list. Finally, all the partitions are then retrieved from the list and a triplet is generated such that no number repeats and the solution is printed.

A generalised algorithm for order n

For a generalized algorithm of order n, we would have to change the number of loops for generating the partitions as the partition size depends on n. The partition size would be (n-2). Additionally, we would have to change the number of variables to retrieve all the values.

Order 5 : Program:- Generating Magic Triangles - Order 5 (Code)

Order 6 : Program:- Generating Magic Triangles - Order 6 (Code)

Order 7 : Program:- Generating Magic Triangles - Order 7(Code)

Reference values for magic triangles

Order	3	4	5	6	7	8	9	10
Number of Magic Triangles	4	18	700	13,123	316,424	7,317,145	176,476,738	4,279,366,371

A very special magic triangle

See Figures 6 and 7. They show a magic triangle of order 4 and magic sum 20, and another triangle with the squares of all the numbers in the original triangle. Amazingly, the second triangle is also a magic triangle (with magic sum 126). This is truly a remarkable occurrence.

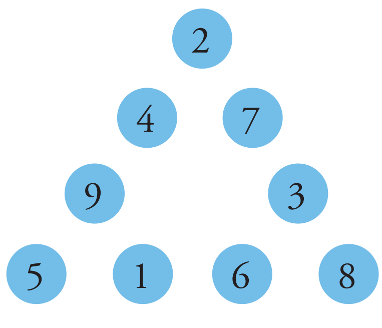

Figure 6. Original magic triangle

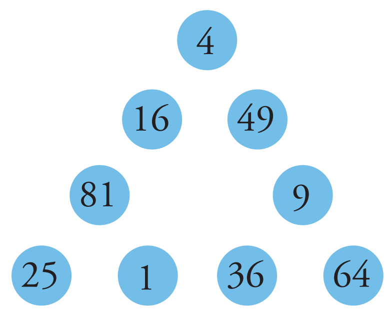

Figure 7. The squared magic triangle

Similarly, there exist magic triangles with different orders which when squared yield another magic triangle. There is 1 such triangle for order 4; 4 such triangles for order 7; and 9 such triangles for order 8. In the case of orders 5 and 6, there are no such triangles.

References

1. Theorem Concerning A Magic Triangle
2. On Adding Magic Triangles
3. Perimeter Magic Triangles

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