Problem Corner

A Conjecture equivalent to the Goldbach Conjecture, and some Consequences

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The *Goldbach conjecture* is one of the oldest and best-known unsolved problems in Number Theory. It states the following:

Conjecture 1 (Goldbach). *Every even natural number greater than 2 can be written as the sum of two prime numbers.*

For example: $6 = 3 + 3$, $8 = 5 + 3$, $10 = 5 + 5$, ..., $2022 = 1009 + 1013$, ...

It was conjectured in the year 1742 by Christian Goldbach.

Closely related to this conjecture is the *Odd Goldbach Conjecture* (or the 3-primes problem):

Conjecture 2 (Odd Goldbach Conjecture). *Every odd integer greater than 5 can be written as the sum of 3 primes.*

For example: $7 = 2 + 2 + 3$, $9 = 3 + 3 + 3$, $11 = 3 + 3 + 5$, ...

It is easy to see that the Goldbach Conjecture implies the Odd Goldbach Conjecture.

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Proof. Consider an odd integer $n \geq 7$. We must show that it can be written as the sum of 3 primes. Consider the even number $n-3$. Since $n-3 \geq 4$, the Goldbach Conjecture (assumed to be true) can be applied. Hence there exist primes p and q such that $n-3 = p+q$. This yields $n = 3+p+q$. We have thus written n as a sum of 3 primes. □

Since the Goldbach Conjecture implies the Odd Goldbach Conjecture, we refer to the Odd Goldbach Conjecture as the *Weak Goldbach Conjecture*.

Here is another easy consequence of the Goldbach Conjecture:

Conjecture 3. *Any square number $n > 4$ can be written as the sum of 3 prime numbers.*

For example: $16 = 2+7+7$, $36 = 2+11+23$, ...

Proof. Consider a square number n. We subdivide the proof into two parts.

Case (i) n is an odd square number greater than 4. In this case, by the previous claim n can be written as the sum of 3 primes.

Case (ii) n is an even square number greater than 4. Let $n = 4m^2$ where m is a positive integer greater than 1. Then $n-2 = 4m^2-2$ is even and greater than or equal to 4. Therefore, by the Goldbach Conjecture, there exist primes p_1 and p_2 such that $n-2 = p_1+p_2$ or $n = 2+p_1+p_2$. We have thus written n as the sum of 3 primes.

We conclude that if the Goldbach conjecture is true, then any square number greater than 4 can be written as the sum of 3 primes. □

We now offer a new conjecture which is equivalent to the Goldbach conjecture.

Conjecture 4 (Golden Conjecture; Sasikumar K)**.** *For any natural number $n > 1$, there exist prime numbers p and q such that $n^2 - pq$ is a square.*

For example: (a) For $n = 4$, we may take $(p,q) = (3,5)$. (b) For $n = 6$, we may take $(p,q) = (5,7)$. (c) For $n = 9$, we may take $(p,q) = (5,13)$.

Theorem 1. The Golden Conjecture and the Goldbach Conjecture are equivalent to each other.

Proof. We shall first show that Goldbach Conjecture $\implies$ Golden Conjecture. So let us suppose that the Goldbach Conjecture is true. Let n be a positive integer greater than 1. Then there exist prime numbers p and q such that $2n = p+q$.

Now consider the quadratic equation $x^2 - 2nx + pq = 0$, i.e.,

$$x^2 - (p+q)x + pq = 0.$$

The roots of this equation are clearly the integers p and q. As the roots are integers, the discriminant must be a square, which means that $4n^2 - 4pq$ is a square. Hence $n^2 - pq$ is a square, which proves the Golden Conjecture.

Next we must show that Golden Conjecture $\implies$ Goldbach Conjecture. So let us suppose that the Golden Conjecture is true. Let $m \geq 4$ be an even integer. Then $m = 2n$ where $n > 1$ is an integer.

By the Golden Conjecture, there exist prime numbers p and q (with, say, $p \geq q$) such that $n^2 - pq$ is a square, say $n^2 - pq = k^2$. We naturally have $0 \leq k < n$.

Consider the quadratic equation $x^2 - 2nx + pq = 0$. Its discriminant is $4n^2 - 4pq = 4k^2$ which is a square, so the equation has integer solutions $n + k$ and $n - k$. Let

$$u = n + k = n + \sqrt{n^2 - pq}, \qquad v = n - k = n - \sqrt{n^2 - pq}.$$

Note that $u + v = 2n$ and $uv = pq$. Also, $u > 1$ (since $n^2 - pq \geq 0$).

We shall show that $v > 1$ too. Since $v > 0$, we have $v \geq 1$. So it is sufficient if we prove that $v \neq 1$.

Suppose that $v = 1$. Then $u = pq$, hence $pq + 1 = 2n$, so $n = \frac{1}{2}(pq + 1)$. Therefore

$$\left(\frac{pq+1}{2}\right)^2 - pq = k^2,$$
$$\therefore \quad (pq-1)^2 = 4k^2,$$
$$\therefore \quad (pq + 2k - 1)(pq - 2k - 1) = 0.$$

Therefore either $pq = -2k + 1$ or $pq = 2k + 1$. Both the possibilities imply that pq is odd. But this contradicts the statement made earlier that $pq + 1 = 2n$, an even number.

We conclude that the Golden Conjecture implies the Goldbach Conjecture. Therefore, the Golden Conjecture and the Goldbach Conjecture are equivalent to one another. □

Conjecture 5 (Maillet). *Every even positive integer can be expressed as the difference of two primes.*

For example: $2 = 5 - 3, \;\; 6 = 11 - 5, \;\; 8 = 11 - 3, \;\; 10 = 13 - 3, \;\; 12 = 17 - 5, \;\; \ldots$.

A very much stronger form of the above conjecture is the following.

Conjecture 6 (de Polignac). *Every even number can be expressed as the difference of two consecutive primes in infinitely many ways.*

For example: $4 = 11 - 7 = 41 - 37 = 83 - 79 = 101 - 97 = \cdots$. Conjecture 6 obviously implies Conjecture 5.

It is remarkable that all these conjectures continue to remain open despite enormous research efforts to prove or disprove them!

References

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