Proof Without Words: Alternating Sum of Odd Numbers

ARNABI SAHA

In this visual proof, we will demonstrate that $\sum_{k=1}^{n}(-1)^{n-k}(2k-1) = n$.

Editor's Note. This is a visual and imaginative, though round-about way of proving this identity. We request our readers to remember that this is just a visualization, but that is the case with many such 'proofs'.

Theorem. We will prove that $\sum_{k=1}^{n}(-1)^{n-k}(2k-1) = n$, where n is a natural number.

Proof. We will provide the visualization of the theorem for $n = 7$ and 6 respectively.

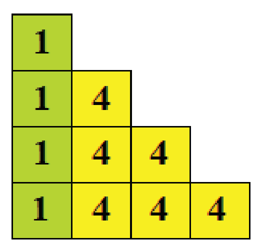
Case-I. First we consider the situation for odd n; for this we show that

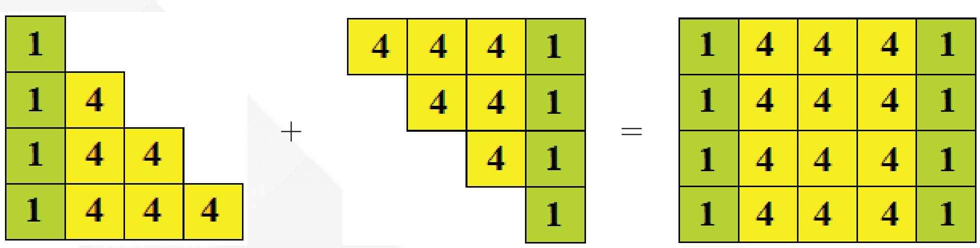
$$1 - 3 + 5 - 7 + 9 - 11 + 13 - \cdots\cdots + (2n-1) = n.$$

For the visualization, we take $n = 7$.

Let,

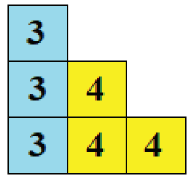
$$\begin{aligned} U &= 1 + 5 + 9 + 13 \\ &= 1 + (1 + (1 \times 4)) \\ &\quad + (1 + (2 \times 4)) \\ &\quad + (1 + (3 \times 4)). \end{aligned}$$

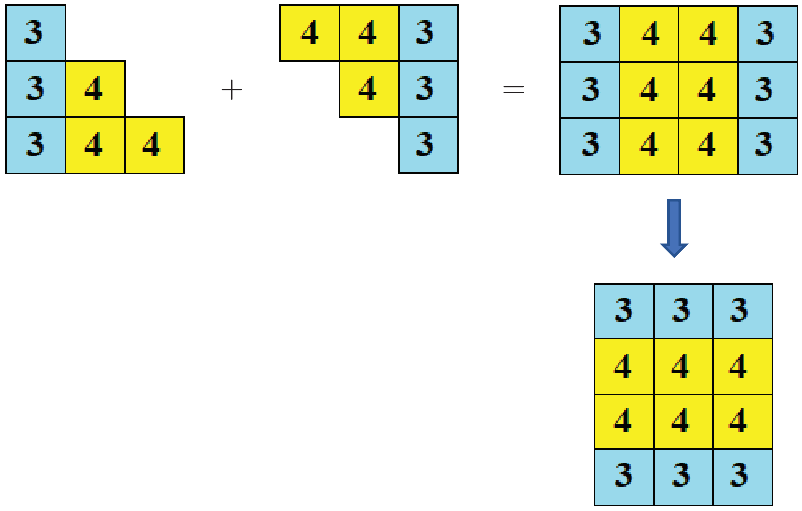
That is $U =$ 

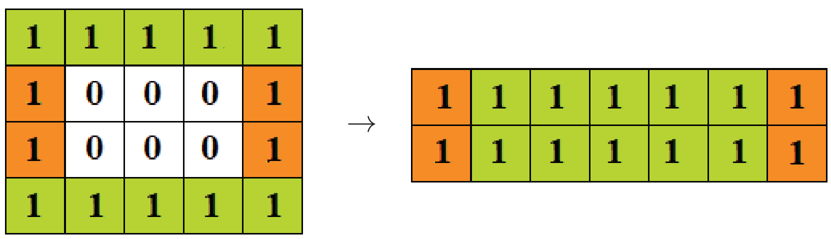
So, $2U =$ 

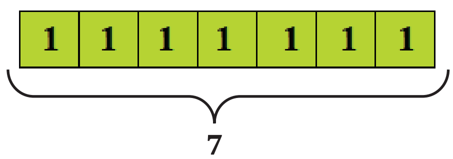
Let,

$$V = 3 + 7 + 11 = 3 + (3 + (1 \times 4)) + (3 + (2 \times 4))$$

Let $V =$ 

That is, $2V =$ 

So, $2U - 2V =$ 

That is, $U - V =$ 

Thus, $1 - 3 + 5 - 7 + 9 - 11 + 13 = 7$.

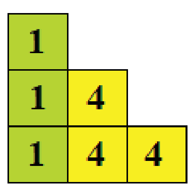
Case-II. Next we consider the situation for even n; for this we show that

$$-1 + 3 - 5 + 7 - 9 + 11 - \cdots\cdots + (2n - 1) = n.$$

For the visualization, we take $n = 6$.

Let

$$U = 1 + 5 + 9 = 1 + (1 + (1 \times 4)) + (1 + (2 \times 4)).$$

That is $U =$ 

That is $2U =$

1		
1	4	
1	4	4

$+$

4	4	1
	4	1
		1

$=$

1	4	4	1
1	4	4	1
1	4	4	1

Let

$$V = 3 + 7 + 11$$
$$= 3 + (3 + (1 \times 4)) + (3 + (2 \times 4)).$$

That is, $V =$

3		
3	4	
3	4	4

That is $2V =$

3		
3	4	
3	4	4

$+$

4	4	3
	4	3
		3

$=$

3	4	4	3
3	4	4	3
3	4	4	3

That is, $2V - 2U =$

2	0	0	2
2	0	0	2
2	0	0	2

So, $V - U =$

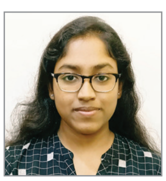
1	1
1	1
1	1

$\}$ 6

Thus, $-1 + 3 - 5 + 7 - 9 + 11 = 6.$

References

1. Chakraborty, B. (2018). Proof without words: The sum of squares. *Mathematical Intelligencer 40* (2), 20.
2. Nelsen, R. (1993). Proofs without Words: Exercise in Visual Thinking. Washington D.C.: The Mathematical Association of America.
3. Nelsen, R. (2015). Proofs without Words III: Further Exercise in Visual Thinking. Washington D.C.: The Mathematical Association of America
4. Sinha, R (2022) Proof Without Words: The Sum Of The First n Odd Integers is a Perfect Square, Ohio Journal of School Mathematics, (Fall 2021).



ARNABI SAHA is a student of Master's degree at the Department of Mathematics, Jadavpur University. Arnabi's professional interests include proof and argumentation, with an emphasis on mathematical techniques that foster stronger conceptual understanding of calculus ideas. More recently, she has become interested in the visualisation of mathematics (in particular, what are known as Proofs Without Words). She may be contacted at arnabisaha99@gmail.com.