ClassRoom

Divisibility by any Odd Number that is not a Multiple of 5

M.R. YADAV &
NARESH KUMAR

Tests of divisibility exist for different divisors. In this article we present a test for divisibility by any odd number that is not a multiple of 5, i.e., divisibility by any number whose one's digit is 1, 3, 7 or 9. Examples of such numbers are 13, 19 and 27.

Basic concepts and notation

- When a number n has one's digit 1, 3, 7 or 9, we can always find a multiple of n whose one's digit is 1 or 9. For example, if $n = 13$, we have $7n = 91$. We also have $3n = 39$.
- When p divides a number a, we denote this by $p|a$.
- If $p|a$ and $p|b$, then $p|(a \pm b)$.
- The greatest common divisor (GCD) of a and b is denoted by $d = (a, b)$.
- If $p|ab$ and $(p, b) = 1$, then $p|a$.

The main result

Given an odd number p whose one's digit is 1, 3, 7 or 9, let $pm = 10q \pm 1$ be a multiple of p whose one's digit is 1 or 9. Let a be the integer which we have to test for divisibility by p. Let b, c be integers such that $0 \leq c \leq 9$ and $a = 10b + c$. Let $d = b \pm qc$ (opposite sign as in the relation for pm). Then $p|a$ if and only if $p|d$.

Keywords: Parity, divisibility

Proof: Case I. When $pm = 10q + 1$.

Let $a = a_n \ldots\ldots a_2 a_1 a_0$ be an $(n+1)$-digit number; then

$$\begin{aligned} a &= a_n 10^n + \ldots + a_1 10^1 + a_0, \\ b &= a_n 10^{n-1} + \ldots + a_2 10^1 + a_1, \\ c &= a_0, \end{aligned}$$

and $d = b - qc$. We must show that $p|a \Leftrightarrow p|d$.

Adding and subtracting $10a_0q$ from a, we obtain

$$\begin{aligned} a &= (a_n 10^n + \ldots + a_1 10^1 + a_0) + 10a_0q - 10a_0q \\ &= \left(a_n 10^n + \ldots + a_1 10^1 - 10a_0q\right) + a_0\,(10q + 1)\,. \end{aligned}$$

The quantity $a_0\,(10q+1)$ is divisible by p. Hence

$$p|a \Leftrightarrow p|\left(a_n 10^n + \ldots + a_1 10^1 - 10a_0q\right).$$

Since $(p, 10) = 1$, it follows that

$$p|\left(a_n 10^n + \ldots + a_1 10^1 - 10a_0q\right) \Leftrightarrow p|\left(a_n 10^{n-1} + \ldots + a_1 - a_0q\right).$$

Now $a_n 10^{n-1} + a_{n-1} 10^{n-2} + \ldots + a_1 = b$ and $a_0 = c$. Hence

$$p\,|a \Leftrightarrow p|\,(b - cq)\,,$$

i.e., $p|a \Leftrightarrow p|d$, as claimed.

Case II. When $P = pm = 10q - 1$. Let $a = a_n \ldots\ldots a_2 a_1 a_0$ be an $(n+1)$-digit number; then

$$\begin{aligned} a &= a_n 10^n + \ldots + a_1 10^1 + a_0, \\ b &= a_n 10^{n-1} + \ldots + a_2 10^1 + a_1, \\ c &= a_0, \end{aligned}$$

and $d = b + qc$. We must show that $p|a \Leftrightarrow p|d$.

Adding and subtracting $10a_0q$ from a, we obtain

$$\begin{aligned} a &= (a_n 10^n + \ldots + a_1 10^1 + a_0) + 10a_0q - 10a_0q \\ &= \left(a_n 10^n + \ldots + a_1 10^1 + 10a_0q\right) - a_0\,(10q - 1)\,. \end{aligned}$$

The quantity $a_0\,(10q-1)$ is divisible by p. Hence

$$p|a \Leftrightarrow p|\left(a_n 10^n + \ldots + a_1 10^1 + 10a_0q\right).$$

Since $(p, 10) = 1$, it follows that

$$p|\left(a_n 10^n + \ldots + a_1 10^1 + 10a_0q\right) \Leftrightarrow p|\left(a_n 10^{n-1} + \ldots + a_1 + a_0q\right).$$

Now $a_n 10^{n-1} + a_{n-1} 10^{n-2} + \ldots + a_1 = b$ and $a_0 = c$. Hence

$$p\,|a \Leftrightarrow p|\,(b + cq)\,,$$

i.e., $p \mid a \Leftrightarrow p \mid d$, as claimed. ■

So, to check divisibility of a by p, we can instead check divisibility of d by p. The result may now be iterated, with d in place of a. Each iteration results in a smaller number to be checked. Ultimately, we are left with a small number for which divisibility by p can be checked mentally.

Remark. When there is a zero at the one's place in a (i.e., if a is divisible by 10), then we may first divide by 10 and then follow the above procedure. (See Problem 3.) This reduces the labour.

Worked out examples

Example 1. Show that 2158 is divisible by 13.

Solution: Here $a = 2158$, so $b = 215$, $c = 8$. Also, $13 \times 3 = 39 = 4 \times 10 - 1$, so $q = 4$.

Next, $d = 215 + (4 \times 8) = 247$. We repeat the procedure with the new number.

$a = 247$, so $b = 24$, $c = 7$, $d = 24 + (4 \times 7) = 52$.

52 is divisible by 13, hence 2158 is divisible by 13. (Check: $2158 \div 13 = 166$.)

Example 2. Show that 65894 is not divisible by 17.

Solution: Here $a = 65894$, so $b = 6589$, $c = 4$. Also, $17 \times 3 = 51 = 5 \times 10 + 1$, so $q = 5$.

So $d = 6589 - (5 \times 4) = 6569$. We repeat the procedure with the new number.

$a = 6569$, so $b = 656$, $c = 9$. Hence $d = 656 - (5 \times 9) = 611$. We repeat the procedure with the new number.

$a = 611$, so $b = 61$, $c = 1$. Hence $d = 61 - (5 \times 1) = 56$.

56 is not divisible by 17, hence 65894 is not divisible by 17. (Check: $65894 \div 17$ yields3876, with remainder 2.)

Example 3. Show that 657894 is not divisible by 27.

Solution: Here $a = 657894$, so $b = 65789$, $c = 4$. Also, $27 \times 3 = 81 = 8 \times 10 + 1$, so $q = 8$.

So $d = 65789 - (8 \times 4) = 65757$. We repeat the procedure with the new number.

$a = 65757$, so $b = 6575$, $c = 7$. Hence $d = 6575 - (8 \times 7) = 6519$. We repeat the procedure with the new number.

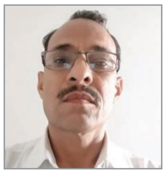
$a = 6519$, so $b = 651$, $c = 9$. Hence $d = 651 - (8 \times 9) = 579$. We repeat the procedure with the new number.

$a = 579$, so $b = 57$, $c = 9$. Hence $d = 57 - (8 \times 9) = -15$.

Since -15 is not divisible by 27, we conclude that 657894 is not divisible by 27 either.

Reference

1. Bharati Krishna Tirthaji Maharaja; *Vedic Mathematics*, Motilal Banarsidass Publishers, 1990.



DR M R YADAV is working as DMS in DEO Office Narnaul. He was Formerly Expert in SCERT Gurugram (Haryana). He is a researcher in Functional Analysis. He may be contacted at mryadavscert@gmail.com

DR NARESH KUMAR is working as an Assistant Professor in DIET Rewari. He is a researcher in Operation Research. He may be contacted at nareshphd08@gmail.com