Three Different Proofs for the Same Task

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A relatively simple and interesting geometric problem is presented, with three "wordless" style proofs attached to it: one in trigonometry and two in geometry. The goal of the task is for the learner to know how to verbally complete the proof according to the drawing and the mathematical expressions. By doing so, they will improve their visual ability to find proofs/solutions for different tasks.

A central part of teaching mathematics is writing proofs and solving problems. These two issues are related to each other. In each of them, the learner must present the method/way of proof or solution by using well-known theorems, attributes, or formulas, while their writing must be continuous and logical, so that the reader can understand the method of solution and confirm its correctness.

However, in reality it is not like that. The writers of mathematics articles, the teachers, as well as the students, usually skip listing some of the steps of the solution, because they think that a part of them are immediately understood and therefore there is no need to detail them. Such situations often leave the reader without an understanding of the solution or its correctness. On the other hand, there are cases where the solution/proof is excessively detailed, down to the smallest details. Such cases tire the reader, and they lose their attention and desire to continue reading.

Proof without words

Especially in the last decade it is possible to see in various journals of mathematical education (some of them prestigious), both as a section and as a decoration, the presentation of "proofs without words" for various tasks. The proofs are usually presented with an illustration/drawing (sometimes with the addition of an auxiliary construction) as well as a listing of mathematical expressions and formulas, but without any words. The reader/student is expected to understand the steps of the proof and be able to formulate them verbally. From this activity, it is expected that the reader/learner will be able to develop their visual proof ability, when a large part of the tasks they face during their mathematics studies are based on tasks with illustrations and mathematical expressions [1-2].

"The illustration (drawing/diagram) and the mathematical expressions, are the lens through which the student understands the proof or identifies the argumentations to the solution."

The "proof without words" is largely like a caricature in which a drawing appears with a certain message (sometimes with 2-5 words), that the viewer has to understand and absorb.

Multiple solutions to the same geometric task

The studies in plane geometry constitute an important element of the studies of Mathematics, due to their contribution to the development of different channels for reasoning. It is important to note that the studies of plane geometry constitute a basis for the studies of trigonometry, solid geometry, analytical geometry, and they also have an important role in other branches of Mathematics. One of the principal objectives of the studies of mathematics is to impart to the students' methods of reasoning that may assist them in other fields of learning and knowledge, and in solutions at higher levels. The meaning of "learning to think" means that the teacher of Mathematics must develop the students' ability to apply information and perform analysis and synthesis at the adequate level of the basic properties, rules and theorems which has been taught at the earlier stages of the teaching process. The solution of different problems is one of the important means for reasoning development. Reasoning development is enhanced by solving problems using several different methods. Finding an additional method of solution using tools from the same mathematical field, and the more so when the tools are from a different field, promotes reasoning development and raises it to a higher level. Implementation of new knowledge in a new situation that leads to a shorter and simpler solution increases the pleasure and satisfaction that one obtains from one's studies in Mathematics.

Integration of fields in problem solution opens a wider view on Mathematics for the students as a comprehensive subject that contains connections and interrelations between the different branches, reveals and accentuates its beauty.

The solution of a problem by a regular method leaves the students indifferent and without any particular reaction. However, a different solution to the same problem may elicit emotional excitement. A special and beautiful solution is surprising, unexpected and in most cases – short and simple [3-4].

The field of problems in plane geometry is a wide field of challenges where many solutions can be found both from this field and from adjacent or related fields.

The problem

Given three identical squares adjacent to each other as seen in Figure 1. In the figure the angles are marked: $\measuredangle HCA = \alpha$, $\measuredangle HDA = \beta$.

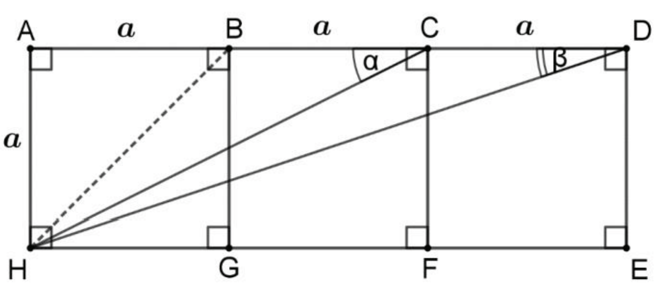

Figure 1

It must be proved that: $\boldsymbol{\alpha + \beta = 45^\circ}$

Proof A: Using trigonometry (Figure 1)

$$\tan\alpha = \frac{a}{2a} = \frac{1}{2}, \tan\beta = \frac{a}{3a} = \frac{1}{3}, \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta},$$

$$\therefore \tan(\alpha+\beta) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1 \Rightarrow \boldsymbol{\alpha+\beta = 45^\circ}.$$

A note

In a classroom activity, it was found that students solved this task using a calculator, which is a legitimate way but less beautiful than the proof presented.

Proof B: Using geometry (Figure 2)

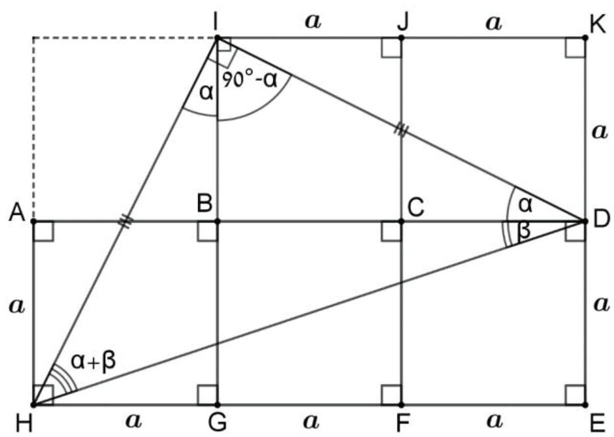

Figure 2

$$\Delta HID: \underbrace{IH{=}ID,\ \measuredangle HID{=}90^\circ,\ \measuredangle IHD{=}\ \measuredangle IDH{=}\alpha{+}\beta,}_{\Downarrow}$$

$$\therefore \alpha+\beta = 45^\circ.$$

Proof C: Using geometry (Figure 3 and 4)

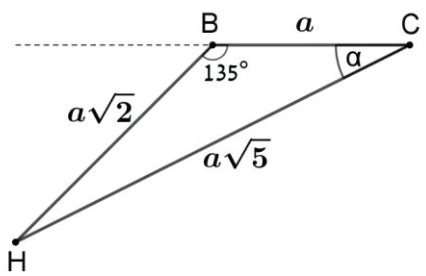

Figure 3

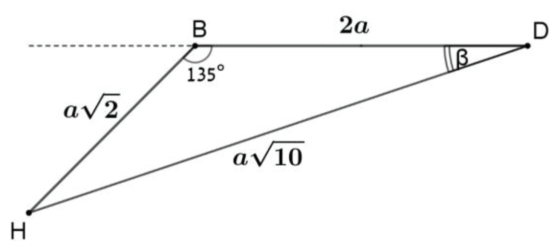

Figure 4

Consider ΔHBC with sides a, $\alpha\sqrt{2}$, $\alpha\sqrt{5}$ and ΔHBD with sides $\alpha\sqrt{2}$, 2a, $\alpha\sqrt{10}$.

$$\frac{\alpha\sqrt{2}}{a} = \frac{2a}{\alpha\sqrt{2}} = \frac{\alpha\sqrt{10}}{\alpha\sqrt{5}} = \sqrt{2} \Rightarrow \Delta CBH \sim \Delta HBD \ \Rightarrow \boldsymbol{\alpha + \beta = 45^\circ}.$$

Notes

1. From the methodical point of view, the students should be expected to know how to complete the verbal reasons at each stage of the proof.
2. For this task there are additional proofs.

Epilogue

From a comprehensive activity with students for teaching mathematics, understanding the reasoning for mathematical tasks presented through illustrations and mathematical expressions is of great importance in the training process. Students acquired another tool for their "mathematical toolbox," and will be able to use it in their classrooms.

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