Detours – Unplanned Learning Opportunities

RUPESH GESOTA

Teachers have often seen that their carefully planned lessons take unexpected turns and end up not achieving their intended learning objectives. Here is a narrative from Rupesh Gesota who seizes such detours as opportunities for unintended learning and formative assessment.

Rupesh's plan was to see relationships between the side lengths in 30-60-90 Triangles. Without revealing this to his students, he told them to construct a couple of 30-60-90 Triangles with hypotenuses 10 cm and 6 cm respectively.

This is a sketch of their findings when he told them to measure the lengths of the remaining sides in both triangles. Rupesh's questions skillfully facilitated the following conclusion from the students.

In both the triangles, the shortest side length is always half the longest one.

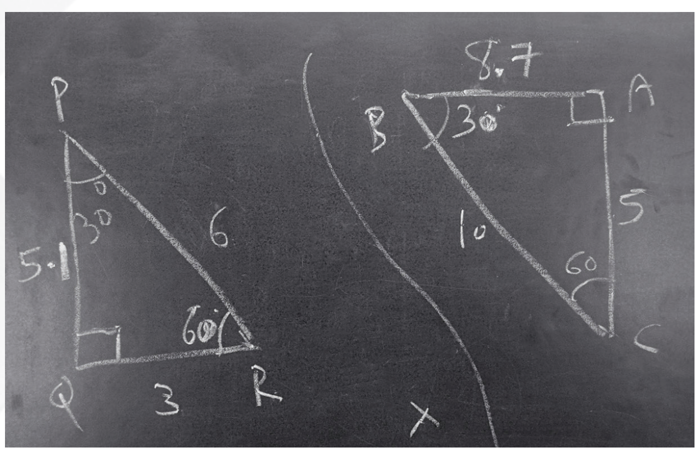

Image 1

Keywords: Triangles, Ratios, Square-Roots, Exploration, Reasoning, Discussion

He then told them to construct two more right triangles, with the hypotenuse as any whole number, but with angles other than 30° & 60° this time. They observed that in these cases the shortest side was not half the hypotenuse (the longest side). So, they said that this relation holds true only when the angles are 30-60-90.

Going back to the 30-60-90 triangle, he asked if there was any relation between the side opposite the 60° angle and the hypotenuse.

Since these side lengths were in decimals (5.1 and 8.7), it was naturally difficult for them to easily relate those to hypotenuse lengths.

He then guided them to the application of Pythagoras theorem, and they reached the step shown in Image 2. (Only the last part of the derivation, dictated by the students and written on the blackboard by Rupesh is shown.)

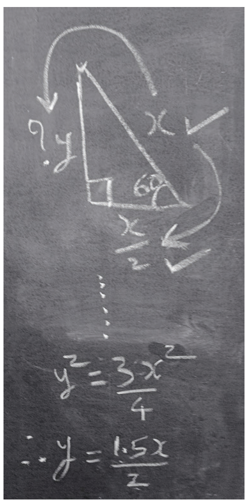

Image 2

Rupesh: How did you arrive at $y = \frac{1.5x}{2}$?

Students: The square root of x^2 is x and since the square root of 1 is 1 and the square root of 4 is 2, the square root of 3 is 1.5.

When asked to verify this, they started multiplying 1.5 by 1.5 using the standard procedure for multiplication of decimals. In Rupesh's words, *"I would have loved seeing them figure out this product mentally through reasoning. (Can you try that way?) I thought we would discuss this approach once they got the answer using this method, but then something else happened."*

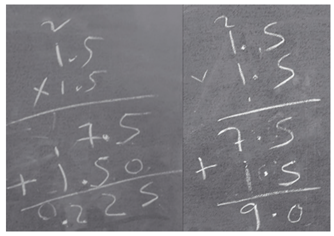

Image 3: "I was now glad that they went by the standard multiplying method. They seemed puzzled / uncomfortable with their results. So, I told them to show their work on the board"

The students realised that both answers were incorrect since they were not between 1 and 4 as the square of 1.5 was expected to be. Rupesh understood that the students had difficulties with decimal multiplication and asked the students for another way to write 1.5.

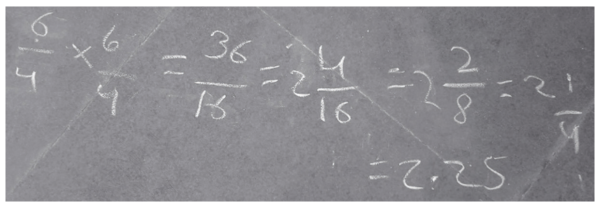

Image 4

They reasoned out that 1.5 = 6/4 because if 6 chapatis are given to 4 people, then each gets one and a half chappati, so 6/ 4 = 1.5. Multiplication of fractions seemed to have been mastered by them and they also reasoned that 1/4 = 0.25 because 1 rupee = 100 paise, so a quarter means 25 paise. They were delighted with this answer 2.25 because it satisfied their expectation that 1.5 × 1.5 needed to be between 1 and 4.

Further, something clicked for the boy who had worked out 1.5 × 1.5 as 0.225 on the board; he went and corrected his answer saying that he had placed the decimal point incorrectly.

After some discussion, the rules of decimal multiplication were arrived at by the students who managed to correct their work based on the rules that they had arrived at. (For a complete account of the facilitation of this discussion, please visit Rupesh's blog, the link is given at the end of the article.) Rupesh gave them plenty of practice in decimal multiplication and when he was confident about their conceptual understanding of the algorithm, he went back to the original question:

Rupesh: What's the value of square root of 3? Is it 1.5?

Students (laughing): No, it will be between 1.5 and 2.

Rupesh: How do you know?

Students: Because we saw that 1.5 squared is 2.25 and 2 squared is 4.

Rupesh set them the task of finding the value of the square root of 3 for homework. And they left happily agreeing to this challenge.

When they arrived the next day, he asked them if they could find the value of the square root of 3. They confessed that they had tried but were not successful. They had arrived at the fact that 1.73 squared gave less than 3 and 1.74 squared gave more than 3 and they had concluded that the value of the square root of 3 was between 1.73 and 1.74.

Rupesh was glad that they worked with numbers having 2 digits after the decimal point, rather than just stopping at 1.7 and 1.8. But then he also wondered why they didn't go beyond that. He told them to list the numbers they had tried for the very first time and they wrote 1.6, 1.7, 1.8 and 1.9.

They also said that in the process of exploration, they had realised that the square root of 3 was between 1.7 and 1.8 and hence tried zooming in on that interval. Rupesh then asked them to write 1.71, 1.72, 1.73, but they stopped at 1.74, saying they did not try beyond this as 1.74 squared crossed 3.... While he agreed with them, he asked them to continue listing beyond 1.74 & they wrote till 1.80. When they confessed that they did not know how to proceed, he drew their attention to the numbers on the left & told them to complete this list too - to write the numbers less than 1.6, and so they wrote till 1.1. [See Image 5]

Rupesh: How did you get these numbers between 1 and 2?

Students: By dividing the range into 10 parts.

Rupesh: Ok… and how did you get the numbers 1.71, 1.72, 1.73, etc.? I am asking this because I don't see them in the first column.

Students: We knew it's between 1.7 & 1.8, so we divided the range 1 to 2 into 100 parts now.

Rupesh circled the two numbers 1.7 and 1.8 when they said this and, while pointing at the circled portions, told them:

"So, can we say this second column is a kind of zoomed-in-picture between 1.7 and 1.8? Numbers which were present but not visible earlier have become visible now because you have divided the range into smaller, i.e., more parts. It's like you have kept a magnifying glass now on the two numbers 1.7 and 1.8."

He paused to help them understand this new perspective, then continued.

"So now you say that the answer, the number, is between 1.73 and 1.74. What can we do now?"

Students: We will divide the range further - into 1000 parts now - so that the numbers between 1.73 and 1.74 become visible. The student who said this also circled the pair 1.73 and 1.74.

They started listing from 1.731, 1.732 and so on till 1.740. So, he asked them what 1.740 represents. They said that it was the same as 1.74. So, then I told them to include another form of 1.73 too because they had circled this number too. Hearing this, the student wrote 1.730 above 1.731 in that column.

Now he drew their attention to the two circled pairs and the list of numbers next to each pair so that they could also actually see and not just visualize that the interval (1.7, 1.8) has the numbers from 1.70 to 1.80 in it and (1.73, 1.74) has the numbers from 1.730 to 1.740 in it.

The picture started looking like Image 5 in some time....

The squares of 1.731, 1.732 and 1.733 were calculated by them manually using the standard algorithm, but when it came to testing the squares of numbers in the next column, (those with more digits after the decimal place), then Rupesh became their assistant and helped them to get the squares of numbers which they wanted, with the help of his phone calculator.

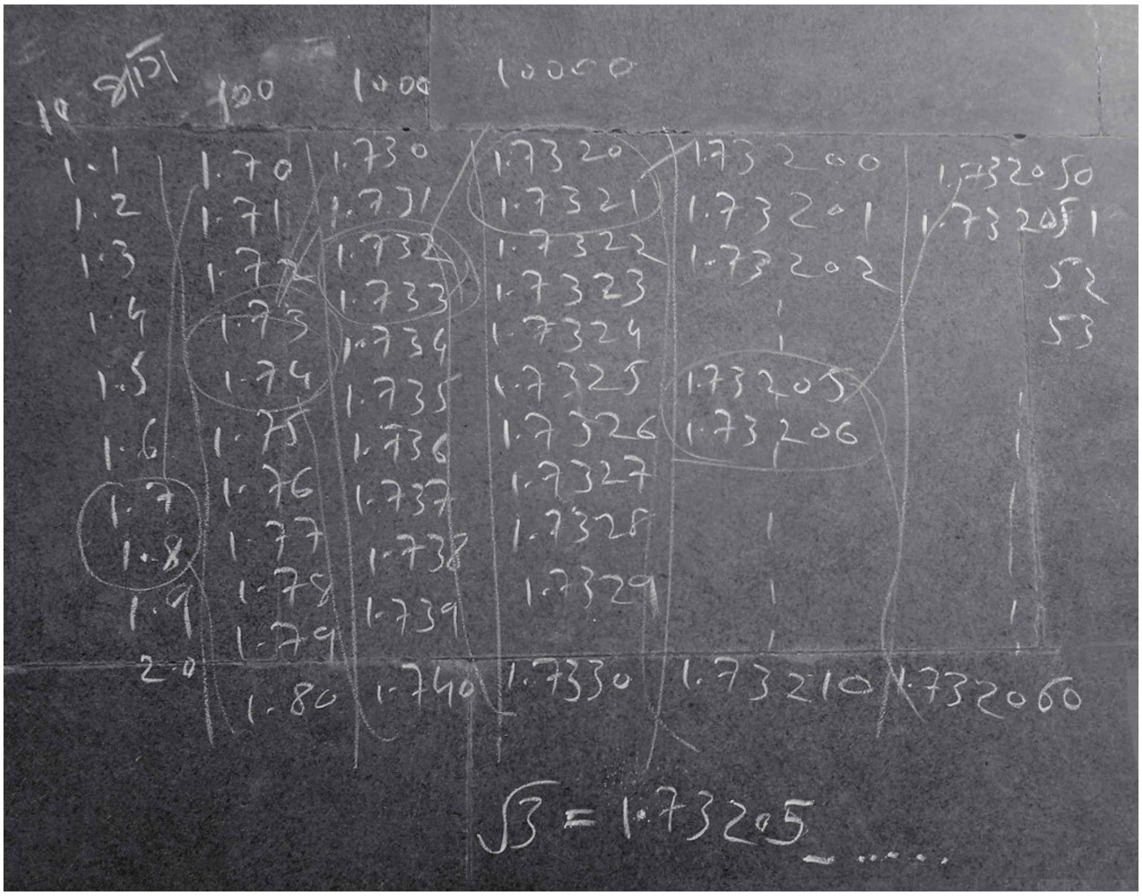

Image 5

Things had gone into auto-pilot mode now and they were enjoying this process, totally surprised as this hunting never seemed to stop, against their expectation. They said that they had never thought that square root of a number (that too such a small one like 3) will have so many digits.

Rupesh also shared with them that they need not list all the numbers in a column but could use dotted lines to indicate these. After some time, he stopped them and asked them what they thought about this process.

Students: Sir, it seems this is never going to stop.... We are just reaching closer and closer to the answer....

Rupesh: How do you know this?

Students: The square of the number comes out to be 2.9999.... or 3.0000 with a few other digits after 9 and 0. And the number of 9's and 0's keep increasing....

He asked them if they could be very sure of at least some digits in the square root of 3?

They looked for a while in all the columns and noticed the growing & unchanging section of digits. As you can see in Image 5, they have written the value of square root of 3 as 1.73205.......

He asked them whether someone who told them that the square root of 3 equals 1.732 was correct.

They replied in the negative and explained that the answer is close to 1.732, but not equal to 1.732.

Looking at their facial expressions and body language, it was clear that this exercise was no less than an adventure ride for them. So now it was time to plug-in this correct value of root 3 into the expression they had arrived at (remember the original question).

Image 6

'x' and *'y'* represent the respective lengths of hypotenuse and the side opposite to the angle measuring 60 degrees in the right-angled triangle.

They had observed that side length opposite to the angle measuring 30° is half the hypotenuse and the next attempt was to find the relation between hypotenuse and side opposite to the angle measuring 60°.

Now, they used the value of $\sqrt{3}$ that they had calculated.

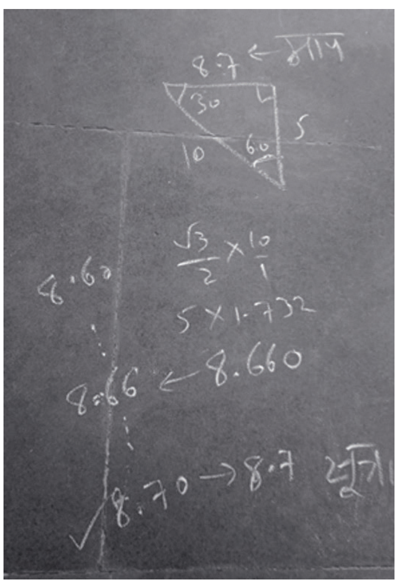

Image 7

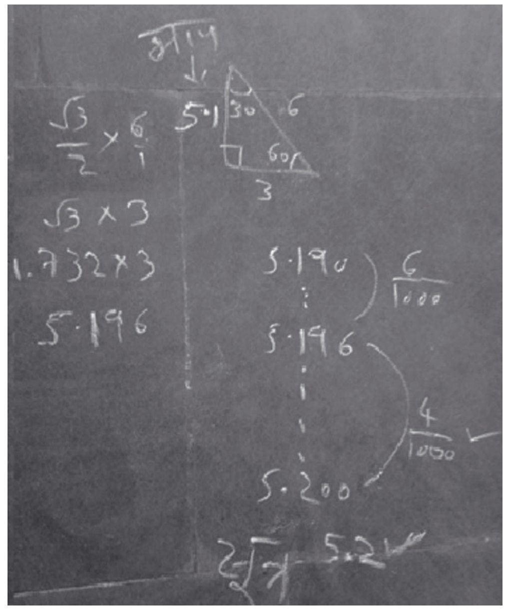

Image 8

They were delighted to see that their measured lengths matched the lengths given by the formula.

We also discussed here about the round-off error, construction/ measurement error, resolution of measuring devices.

"Now we know why the square root 3 is written just as $\sqrt{3}$ in textbooks!" they exclaimed.

So, what about the square root of 2?

They thought for a second and understood there can be many such numbers - square root of 5, square root of 7, etc., and I concluded this discussion by saying that such numbers are called as Irrational Numbers (of course with this thought in mind that this definition / explanation is not yet so precise and complete yet).

I moved to another topic, while giving them this assignment to find the value of square root of 2 (another irrational number which appears much in school mathematics) and they happily agreed to work on this.

For a complete account of this discussion visit

http://rupeshgesota.blogspot.com/2023/02/whats-value-of-square-root-of-3-part-1.html

https://rupeshgesota.blogspot.com/2023/02/whats-value-of-square-root-of-3-part-2.html?m=1

RUPESH GESOTA is an engineer-turned-mathematics-educator. He loves to see the sparkle of understanding in the eyes of children, and he finds it inspiring to realise that he was part of this enlightenment process. He also loves working with their parents and teachers to make the process of Math education meaningful as well as joyful. To read more of his classroom stories check his blog www.rupeshgesota.blogspot.com. Rupesh may be reached through his website www.rupeshgesota.weebly.com