Features

Two New Proofs of the Pythagorean Theorem - Part I

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The Pythagorean theorem ('PT' for short) is easily the best known result in all of mathematics. What is less well-known is the fact that among all theorems in mathematics, it holds the "world record" for the number of different proofs. There is no other theorem that even comes close! (See [2] and [3].) In the book *The Pythagorean Proposition* [1] (published in 1940), the author Elisha S. Loomis lists as many as 370 different proofs of the theorem. Since that time, close to a century back, still more proofs have appeared.

We present two of these proofs in this two-part article. The first one is by two high-school teenagers, Calcea Johnson and Ne'Kiya Jackson, both at St. Mary's Academy in New Orleans, USA. It has not yet been published so what we present here has been gleaned from media reports on their proof; see [4], [5], [6] and [7]. The second is an adaptation of a proof [8] by Professor Kaushik Basu, a well-known World Bank economist; he describes the proof as "new and very long" but gives a poetic and eloquent justification for adding this proof to the long list of existing proofs. This will appear in Part II of the article.

The proof by Calcea Johnson & Ne'Kiya Jackson raises an extremely interesting question. *Can there be a proof of the PT based on trigonometry?* In [1], the author claims definitively and strongly that there cannot be such a proof. It turns out that Loomis was wrong in making this claim, and the proof by Johnson & Jackson is itself a counterexample! We will say more about this interesting debate below.

Keywords: Pythagorean theorem, Kaushik Basu, Calcea Johnson, Ne'Kiya Jackson, St. Mary's Academy, New Orleans, trigonometric proof

Proof by Calcea Johnson and Ne'Kiya Jackson

We start by posing a basic question: *Can there be a trigonometric proof of the PT?*

It is possible that by 'trigonometric proof' we have in mind the following 'argument.' Let $\triangle ABC$ be given, right-angled at C. Using the usual symbols we must prove that $c^2 = a^2 + b^2$. Observe that $\sin A = a/c$ and $\cos A = b/c$. Since $\sin^2 A + \cos^2 A = 1$, we obtain $a^2/c^2 + b^2/c^2 = 1$, or $c^2 = a^2 + b^2$. Hence proved!

But how do we know that $\sin^2 A + \cos^2 A = 1$? By using the Pythagorean Theorem, of course! So we have got trapped into a circular argument here, which means that we have not actually proved anything.

But is there a way of proving the basic trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ without using the PT and using only the definitions of sine and cosine? Yes, there is: we simply mimic the proof-by-similar-triangles of the PT. Here are the details (Figure 1). Let $\triangle ABC$ be right-angled at vertex C; let its hypotenuse AB have unit length. Denote $\measuredangle ABC$ by θ. Draw a perpendicular CD from C to AB. Then $\measuredangle ACD = \theta$.

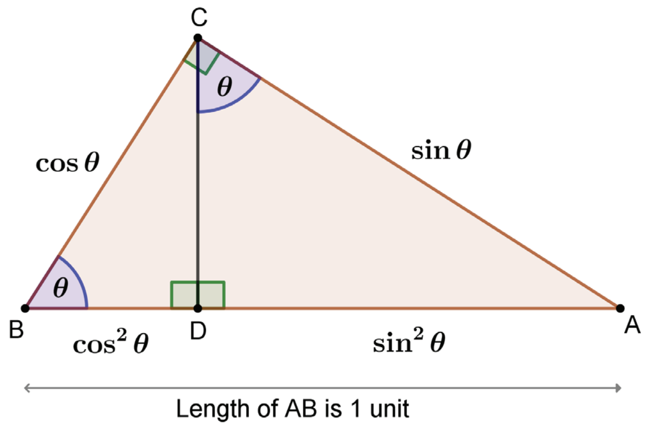

Figure 1. Proof of the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ and the PT.

Since $AB = 1$, we get $CA = \sin\theta$ and $CB = \cos\theta$, from the definitions of the trigonometric ratios. Since $\triangle CBD$ is right-angled with hypotenuse CB, we have

$$\frac{BD}{CB} = \cos\theta, \qquad \therefore\ BD = \cos^2\theta. \tag{1}$$

Since $\triangle CAD$ is right-angled with hypotenuse AC, we have

$$\frac{AD}{CA} = \sin\theta, \qquad \therefore\ AD = \sin^2\theta. \tag{2}$$

Since $BD + AD = 1$, we get $\sin^2\theta + \cos^2\theta = 1$, as required. And since $\cos\theta = BC/AB$ and $\sin\theta = AC/AB$, we get $CB^2 + CA^2 = AB^2$ or $a^2 + b^2 = c^2$, which is the PT. This way, we arrive at the statements of the PT and the basic trigonometric identity ($\sin^2\theta + \cos^2\theta = 1$) at the same time. *Now there is no circularity of argument.*

With that preamble, let us study the proof given by Ms. Calcea Johnson and Ms. Ne'Kiya Jackson of New Orleans; see Figure 2.

Ne'Kiya Jackson, left, and Calcea Johnson recently presented their findings at the American Mathematical Society's south-eastern chapter's semi-annual meeting. Photograph: WWL-TV

Figure 2. Calcea Johnson and Ne'Kiya Jackson of St. Mary's Academy, New Orleans, USA. Credit: https://www.theguardian.com/us-news/2023/mar/24/new-orleans-pythagoras-theorem-trigonometry-prove?CMP=oth_b-aplnews_d-1

Here are the prerequisites needed to understand the proof.

(1) The basic theorems concerning similar triangles (the results are well-known so we do not list them here);

(2) The formula for the limiting sum of an infinite geometric progression with first term a and common ratio r between -1 and 1:

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \qquad (-1 < r < 1);$$

(3) The double-angle trigonometric identity $\sin 2\theta = 2\sin\theta \cdot \cos\theta$.

Figure 3 shows the given $\triangle ABC$ which is right-angled at C. We must show that $c^2 = a^2 + b^2$. We shall assume throughout that $a < b$; equivalently, that $\alpha < \beta$.

Reflect $\triangle ABC$ in AC to give $\triangle ADC$; then $\measuredangle BAD = 2\alpha$. Since $\alpha < \beta$, it follows that $\alpha < 45°$, so $\measuredangle BAD < 90°$. Draw ray AD. Draw a line through B perpendicular to AB. This line will meet ray AD at M, say. (The two lines meet since $\measuredangle BAM < 90°$.) Join BM. Now draw $DE \perp BD$, meeting BM at E; draw $EF \parallel BD$, meeting ray AM at F; draw $FG \perp BD$, meeting BM at G; draw $GH \parallel BD$, meeting ray AM at H; and so on . . . (see the 'etc' in Figure 3).

We shall now compute the lengths of BM and AM; this yields the desired result. We shall feel free to make use of trigonometric identities — *provided that those identities do not require the PT for their proofs.*

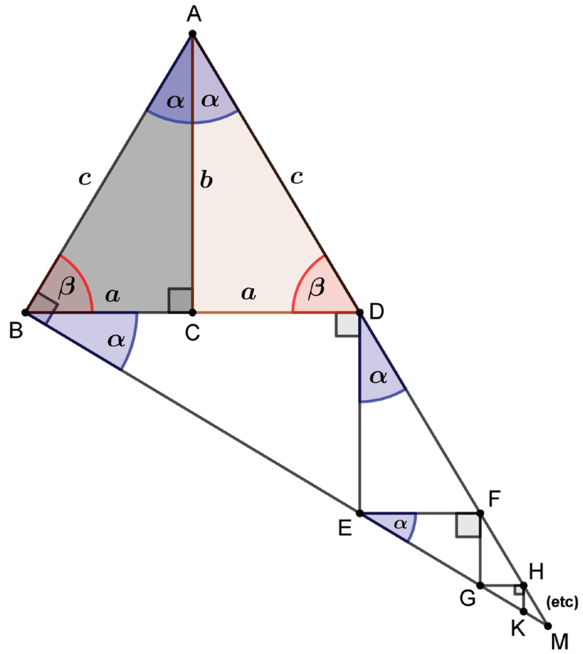

Figure 3. Proof of the PT by Calcea Johnson and Ne'Kiya Jackson.

Note the many triangles that are similar to $\triangle ABC$. We have:

$$\triangle ABC \sim \triangle BED \sim \triangle DFE \sim \triangle EGF \sim \triangle FHG \sim \triangle GKH \sim \cdots, \tag{3}$$

as each of these triangles has angles α, β, 90°.

It follows that $DE : BD : BE = a : b : c$, so $DE : 2a : BE = a : b : c$, hence

$$BE = \frac{2ac}{b}, \qquad DE = \frac{2a^2}{b}. \tag{4}$$

Next: $EF : DE : DF = a : b : c$, so $EF : 2a^2/b : DF = a : b : c$, hence

$$EF = \frac{2a^3}{b^2}, \qquad DF = \frac{2a^2c}{b^2}. \tag{5}$$

Next: $FG : EF : EG = a : b : c$, so $FG : 2a^3/b^2 : EG = a : b : c$, hence

$$FG = \frac{2a^4}{b^3}, \qquad EG = \frac{2a^3c}{b^3}. \tag{6}$$

Proceeding thus, we find that $DF, FH, \ldots$ form a geometric progression with common ratio a^2/b^2:

$$DF = c \cdot \frac{2a^2}{b^2}, \quad FH = c \cdot \frac{2a^4}{b^4}, \quad \ldots. \tag{7}$$

Similarly, $BE, EG, GK, \ldots$ form a geometric progression with the same common ratio, a^2/b^2:

$$BE = c \cdot \frac{2a}{b}, \quad EG = c \cdot \frac{2a^3}{b^3}, \quad GK = c \cdot \frac{2a^5}{b^5}, \quad \ldots. \tag{8}$$

We are now able to compute the total length of the segments DF, FH, ... by summing an infinite geometric progression with common ratio less than 1:

$$\begin{aligned} DF + FH + \cdots &= c \cdot \frac{2a^2}{b^2} \cdot \left(1 + \frac{a^2}{b^2} + \frac{a^4}{b^4} + \frac{a^6}{b^6} + \cdots\right) \\ &= c \cdot \frac{2a^2}{b^2} \cdot \frac{1}{1 - a^2/b^2} = \frac{2ca^2}{b^2 - a^2}, \end{aligned} \tag{9}$$

and therefore:

$$AM = c + \frac{2ca^2}{b^2 - a^2} = c \cdot \frac{a^2 + b^2}{b^2 - a^2}. \tag{10}$$

Similarly,

$$\begin{aligned} BM &= BE + EG + GK + \cdots \\ &= \frac{2ac}{b} \cdot \left(1 + \frac{a^2}{b^2} + \frac{a^4}{b^4} + \frac{a^6}{b^6} + \cdots\right) \\ &= \frac{2ac}{b} \cdot \frac{1}{1 - a^2/b^2} = \frac{2abc}{b^2 - a^2}. \end{aligned} \tag{11}$$

It follows that

$$\frac{BM}{AM} = \left(\frac{2abc}{b^2 - a^2}\right) \div c\left(\frac{a^2 + b^2}{b^2 - a^2}\right) = \frac{2ab}{a^2 + b^2}. \tag{12}$$

On the other hand, $BM/AM = \sin 2\alpha$, as may be seen by observing the angles of the right-angled triangle ABM. Hence:

$$\sin 2\alpha = \frac{2ab}{a^2 + b^2}. \tag{13}$$

We now make use of the trigonometric identity $\sin 2\alpha = 2 \sin \alpha \cos \alpha$. From $\triangle ABC$, we get

$$\sin \alpha = \frac{a}{c}, \qquad \cos \alpha = \frac{b}{c}, \qquad \therefore \ \sin 2\alpha = \frac{2ab}{c^2}. \tag{14}$$

From the two relations for $\sin 2\alpha$, we conclude that

$$c^2 = a^2 + b^2, \tag{15}$$

and there we have it: the Pythagorean theorem! □

You may wonder about the identity $\sin 2\alpha = 2 \sin \alpha \cos \alpha$. Does it require the PT for its proof? No, it does not. (Homework for the reader!) So there is no circular reasoning in this proof.

You may also wonder what happens to the proof in the isosceles case when $a = b$ (or $\alpha = \beta$). We leave this part too to the reader.

Closing remark. Though the proof by Calcea Johnson and Ne'Kiya Jackson is long and involved, it is also subtle and deep. A truly wonderful achievement by the two teenagers.

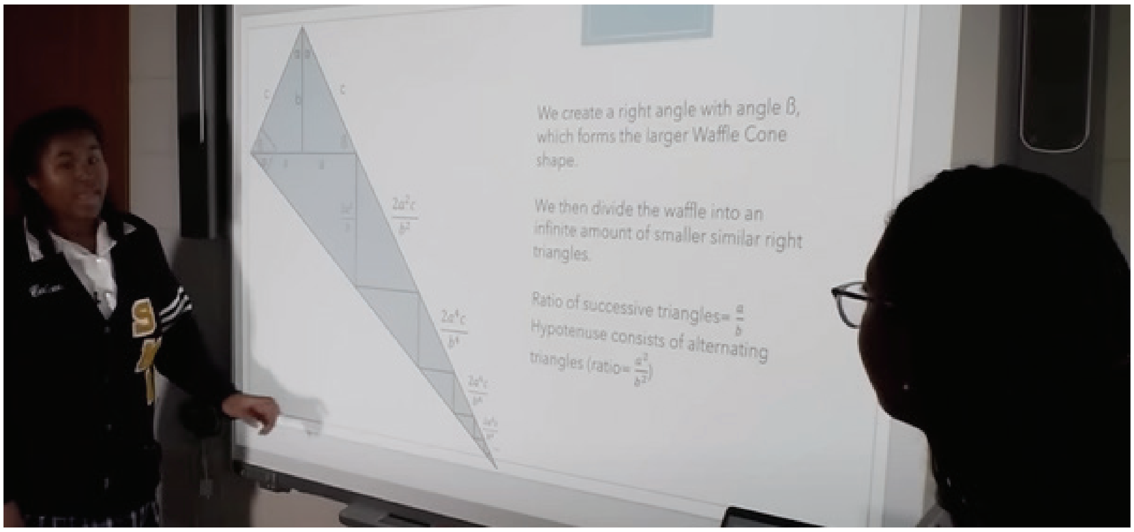

Figure 4. Calcea Johnson and Ne'Kiya Jackson presenting their proof.

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