Menelaus's Theorem

ANKUSH KUMAR PARCHA

Menelaus' theorem is an extremely important result in higher Euclidean geometry. We offer a proof of the theorem here using the sine rule from trigonometry.

Menelaus's Theorem

In ΔABC a transversal line crosses the side lines CA, AB and BC at points Q, P and M, respectively. Then we have the following equality:

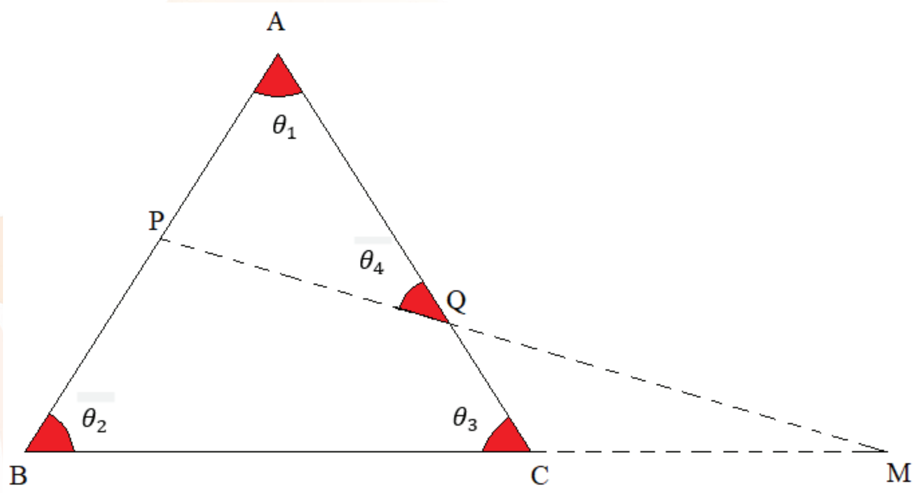
$$\frac{BM}{MC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = 1.$$

Proof

Apply the sine law in ΔAPQ:

$$\frac{\sin\theta_1}{PQ} = \frac{\sin\theta_4}{AP} = \frac{\sin\left(180^\circ - \theta_1 - \theta_4\right)}{QA},$$

$$\text{therefore } \frac{\sin\theta_4}{AP} = \frac{\sin\left(\theta_1 + \theta_4\right)}{QA} \implies \frac{AP}{QA} = \frac{\sin\theta_4}{\sin\left(\theta_1 + \theta_4\right)}.$$



Keywords: Euclidean geometry, Menelaus, proof, sine rule.

Again, applying the sine law in ΔQCM:

$$\frac{\sin\theta_3}{QM} = \frac{\sin\theta_4}{CM} = \frac{\sin(180^\circ + \theta_3 - \theta_4)}{CQ},$$
$$\text{therefore } \frac{\sin\theta_4}{CM} = \frac{\sin(\theta_3 - \theta_4)}{CQ} \implies \frac{CQ}{CM} = \frac{\sin(\theta_3 - \theta_4)}{\sin\theta_4}.$$

Again, applying the sine law in ΔPBM:

$$\frac{\sin(180^\circ + \theta_3 - \theta_4)}{PB} = \frac{\sin(\theta_1 + \theta_4)}{BM} = \frac{\sin\theta_{2,}}{PM}$$
$$\text{therefore } \frac{\sin(\theta_3 - \theta_4)}{PB} = \frac{\sin(\theta_1 + \theta_4)}{BM} \implies \frac{BM}{PB} = \frac{\sin(\theta_1 + \theta_4)}{\sin(\theta_3 - \theta_4)}.$$

Multiplying the corresponding sides of the three equalities, we get:

$$\frac{AP}{QA} \cdot \frac{CQ}{CM} \cdot \frac{BM}{PB} = \frac{\sin\theta_4}{\sin(\theta_1 + \theta_4)} \cdot \frac{\sin(\theta_3 - \theta_4)}{\sin\theta_4} \cdot \frac{\sin(\theta_1 + \theta_4)}{\sin(\theta_3 - \theta_4)},$$
$$\text{therefore } \frac{BM}{MC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = 1.$$

Corollary

$$\left(1 + \frac{AP}{PB}\right)\left(1 + \frac{CM}{CB}\right) = \frac{QM}{QP} \cdot \frac{QA}{QC}.$$

Proof

For ΔMBP with AQC as transversal, Menelaus's Theorem can be written as

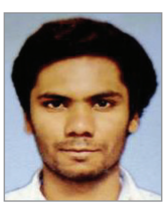
$$\frac{BA}{AP} \cdot \frac{PQ}{QM} \cdot \frac{MC}{CB} = 1.$$

Multiplying this with the previous result, we get:

$$\frac{BA}{AP} \cdot \frac{PQ}{QM} \cdot \frac{MC}{CB} \cdot \frac{BM}{MC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = 1,$$
$$\text{therefore } \frac{BA}{PB} \cdot \frac{BM}{CB} = \frac{QM}{QP} \cdot \frac{QA}{QC} \implies \frac{BP + PA}{PB} \cdot \frac{BC + CM}{CB} = \frac{QM}{QP} \cdot \frac{QA}{QC}.$$

Or:

$$\left(1 + \frac{AP}{PB}\right)\left(1 + \frac{CM}{CB}\right) = \frac{QM}{QP} \cdot \frac{QA}{QC}.$$



ANKUSH KUMAR PARCHA is currently studying BSc Physics Honours (second year) at Indira Gandhi National Open University (IGNOU), New Delhi. He has a deep interest in doing research in Physics and Mathematics. He has contributed many articles, problems, and solutions to magazines such as *Mathematical Gazette*, *AMJ*, *ISROSET*, *SSMJ*, *Pentagon*, *Octagon*, *At Right Angles* and *Mathematical Reflections*. He may be contacted at ankushparcha11@gmail.com.