Student Corner

A Problem from the Putnam 2022 Competition

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In this article, we discuss a problem on polynomials adapted from the Putnam competition of 2022.

Problem 1. Let n be an integer with $n \geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients in $(p(x))^2$?

Developing the strategy. To develop an insight into the solution, we study the cases $n = 2$ and $n = 3$. We then use our observations to conjecture a bound for the number of negative coefficients in $(p(x))^2$, and we then prove the conjecture by a general argument. We finally construct a polynomial $p(x)$ of degree n with the largest possible number of negative coefficients in $(p(x))^2$.

Solution. First, observe that the coefficients of x^n and x^0 in $(p(x))^2$ are always positive. Hence, in order to maximize the number of negative coefficients in $(p(x))^2$, let us see if it is possible for all the remaining coefficients to be negative.

Keywords: Putnam competition, polynomials, coefficient, mathematical induction

Consider the case for $n = 2$. Let

$$p(x) = a_2x^2 + a_1x + a_0,$$

where, without any loss of generality, we may assume that $a_0 \geq 0$, because $(p(x))^2 = (-p(x))^2$. Then

$$(p(x))^2 = \left(a_2x^2 + a_1x + a_0\right)^2 = a_2^2x^4 + (2a_1a_2)x^3 + (2a_0a_2 + a_1^2)x^2 + 2a_0a_1x + a_0^2.$$

The coefficients of x^4 and x^0 are non-negative. Let us see if the coefficients of x, x^2, x^3 can all be negative.

If the coefficient of x is negative, i.e., $2a_0a_1 < 0$, then as $a_0 \geq 0$, it follows that $a_1 < 0$.

If the coefficient of x^2 is negative, i.e., $2a_0a_2 + a_1^2 < 0$, then it follows that $a_2 < 0$. But then, the coefficient of x^3 will be positive, since $2a_1a_2 > 0$.

We observe that not all of the coefficients of x, x^2, x^3 can be negative.

Therefore, for $n = 2$, the number of negative coefficients in $(p(x))^2$ cannot exceed 2.

Next, consider case $n = 3$. Let

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0,$$

with $a_0 \geq 0$. Then

$$\begin{aligned}(p(x))^2 &= \left(a_3x^3 + a_2x^2 + a_1x + a_0\right)^2 \\ &= a_3^2x^6 + 2a_3a_2x^5 + \left(a_2^2 + 2a_3a_1\right)x^4 + \left(2a_2a_1 + 2a_3a_0\right)x^3 \\ &\quad + \left(2a_0a_2 + a_1^2\right)x^2 + 2a_0a_1x + a_0^2.\end{aligned}$$

If coefficient of x is negative, i.e., $2a_0a_1 < 0$, then $a_1 < 0$.

If the coefficient of x^2 is negative, i.e., $(2a_0a_2 + a_1^2) < 0$, then $2a_0a_2 < 0$. As $a_0 \geq 0$, we must have $a_2 < 0$. Now, if the coefficient of x^3 is negative, i.e., $2a_2a_1 + 2a_3a_0 < 0$, then we must have $a_3 < 0$. Thus, we have $a_2 < 0, a_3 < 0$, so $2a_2a_3 > 0$. We see that the coefficients of x^4 and x^5 are positive.

So not all the coefficients of x, x^2, x^3, x^4, x^5 can be made negative.

Therefore, for $n = 3$, the number of negative coefficients in $(p(x))^2$ cannot exceed 4.

Based on these observations, we make a guess that for a polynomial $p(x)$ of degree n, the number of negative coefficients in $(p(x))^2$ cannot exceed $2n - 2$.

We establish this claim in the next section.

Bound for the number of negative coefficients in $(p(x))^2$. Since the coefficients of x^{2n} and x^0 in $(p(x))^2$ are always positive, it suffices to show that we cannot have every other coefficient negative.

Suppose that this is the case; i.e., the coefficients of x^{2n-1}, x^{2n-2}, ..., x^2, x^1 are all negative.

Write $p(x) = a_nx^n + \cdots + a_1x + a_0$; without loss of generality, let $a_0 > 0$. We start by proving, using induction, that a_1, a_2, ..., a_n must all be negative.

The base case of $a_1 < 0$ is true because otherwise the coefficient of x in $(p(x))^2$ would be non-negative.

Now suppose that $a_1 < 0$, $a_2 < 0$, $a_3 < 0$, ..., $a_k < 0$ for some value of k with $1 \leq k \leq n - 1$. We shall prove that $a_{k+1} < 0$ as well.

Let the coefficient of x^{k+1} in $(p(x))^2$ be b_{k+1}; then

$$b_{k+1} = 2a_0a_{k+1} + a_1a_k + a_2a_{k-1} + \cdots + a_ka_1.$$

Therefore,

$$2a_0a_{k+1} = b_{k+1} - (a_1a_k + a_2a_{k-1} + \ldots + a_ka_1).$$

By the inductive hypothesis, the summation in the bracket is positive, and by assumption,

$$b_{k+1} < 0.$$

As $a_0 > 0$, it follows that $a_{k+1} < 0$, completing the induction.

It follows that $a_1, a_2, \ldots, a_n$ are all negative.

But if $a_1, a_2, \ldots, a_n < 0$, then the coefficient of x^{2n-1} in $(p(x))^2$ (which equals $2a_{n-1}a_0$) must be positive. Thus, we have a contradiction to the assumption that the coefficients of all the terms (x^{2n-1} through x^1) of $(p(x))^2$ are all negative.

This shows that the number of negative coefficients in $(p(x))^2$ is $\leq 2n-2$.

Construction of an optimal polynomial. We now show that there is a polynomial $p(x)$ of degree n, with real coefficients, such that the number of negative coefficients in $(p(x))^2$ is $2n-2$.

That is, we show that there is a polynomial for which the bound proved above is attained.

To this end we consider:

$$p(x) = x^n - ax^{n-1} - ax^{n-2} - \cdots - ax^2 - ax + 1, \quad \text{where } a > 0.$$

We have not yet specified the value of a but we shall do so shortly. The above expression may be written as

$$p(x) = (x^n + 1) - a\left(x^{n-1} + x^{n-2} + \cdots + x\right).$$

Then,

$$\begin{aligned}
(p(x))^2 &= \left(x^{2n} + 2x^n + 1\right) + a^2\left(x^{n-1} + x^{n-2} + \cdots + x\right)^2 \\
&\quad - 2a\left(x^n + 1\right)\left(x^{n-1} + x^{n-2} + \cdots + x\right) \\
&= \left(x^{2n} + 2x^n + 1\right) + a^2\left(x^{2n-2} + 2x^{2n-3} + 3x^{2n-4} + \cdots + (n-1)x^n\right. \\
&\quad \left. + (n-2)x^{n-1} + (n-3)x^{n-2} + \cdots + x^2\right) \\
&\quad\quad - 2a\left(x^{2n-1} + x^{2n-2} + \cdots + x^{n+1} + x^{n-1} + x^{n-2} + \cdots + x\right) \\
&= x^{2n} + (-2a)x^{2n-1} + \left(a^2 - 2a\right)x^{2n-2} + \cdots + \left((n-2)a^2 - 2a\right)x^{n+1} \\
&\quad + \left(2 + (n-1)a^2\right)x^n + \left((n-2)a^2 - 2a\right)x^{n-1} + \cdots + (a^2 - 2a)x^2 + (-2a)x + 1.
\end{aligned}$$

Now, if we choose $a > 0$ such that $2a > (n-2)a^2$ or, equivalently, such that

$$0 < a < \frac{2}{n-2},$$

then the coefficients of x^{2n}, x^n, x^0 are positive and all the remaining coefficients are negative.

Thus, we have a polynomial $p(x)$ of degree n such that the number of negative coefficients in $(p(x))^2$ is $2n-2$.

We conclude that among all real polynomials $p(x)$ of degree n, the largest possible number of negative coefficients in $(p(x))^2$ is $2n-2$. □

We leave the following problem to the reader, as a challenge.

Problem 2. Let T denote the set of all polynomials with real coefficients of degree n such that all roots are real. As $p(x)$ varies over T, what is the maximum number of negative coefficients in $(p(x))^2$?

References

1. William Lowell Putnam Mathematical Competition Problems, https://www. maa. org/math-competitions/putnam-competition

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