# Real World Implications of a KVPY Problem

MAHIT WARHADPANDE & ABHRONEEL GHOSH

Abstract: Where do the intriguing problems seen in puzzle books and competitive exams come from? Are they mere mathematical curiosities or do they represent something in the real world? If the latter, then does our awareness of the real-world connection provide any deeper insights?

## Introduction

The following question (see Figure 1) appeared in the Kishore Vaigyanik Protsahan Yojana (KVPY) 2017 SX/SB question paper [1].

*17) Consider the following parametric equation of a curve:*

$$x(\theta) = |\cos 4\theta|\cos\theta;\ y(\theta) = |\cos 4\theta|\sin\theta;\ \text{ for } 0 \leq \theta \leq 2\pi$$

*Which one of the following graphs represents the curve?*

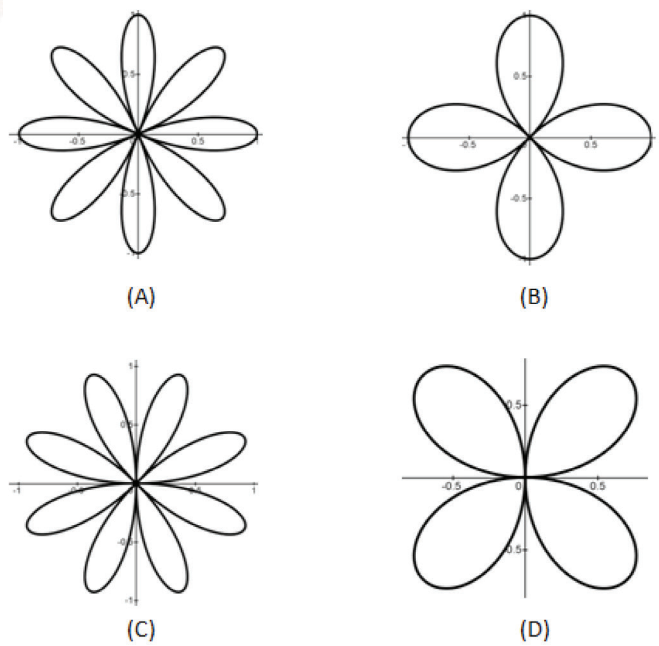


Figure 1. Question 17 in KVPY 2017 for SX/SB stream.

*Keywords: KVPY, retrograde motion, Desmos, Geometric Chuck.*

One common approach to solving this problem would be to set $\theta = \pi/2$ so that $x(\theta) = 0$ and $y(\theta) = 1$. This eliminates options (C) and (D). To choose between (A) and (B) we could make another "intelligent" guess and set $\theta = \pi/4$ so that $x(\theta) = 1/\sqrt{2} = y(\theta)$. This eliminates option (B) since it does not have any point on the line $y = x$ other than $(0, 0)$. End of problem.

Or is it? Why would anyone come up with equations or graphs like this? Do they describe any real phenomenon or are they purely mathematical constructs whose sole application is as examination questions? Here is an account of how we were able to relate this particular question to a real-world phenomenon. This not only helped us answer the KVPY question (well, almost!), but also allowed us to understand several properties of these curves and equations through the behaviour of the corresponding physical system and vice-versa. We used the Desmos graphing calculator [3] to study the graphs produced. See the Appendix for directions on using this software.

### Apparent Planetary Motion

Figure 2 shows a simplified heliocentric model for planetary motion: all planetary orbits are circular and all planets start at zero phase in their respective orbits.

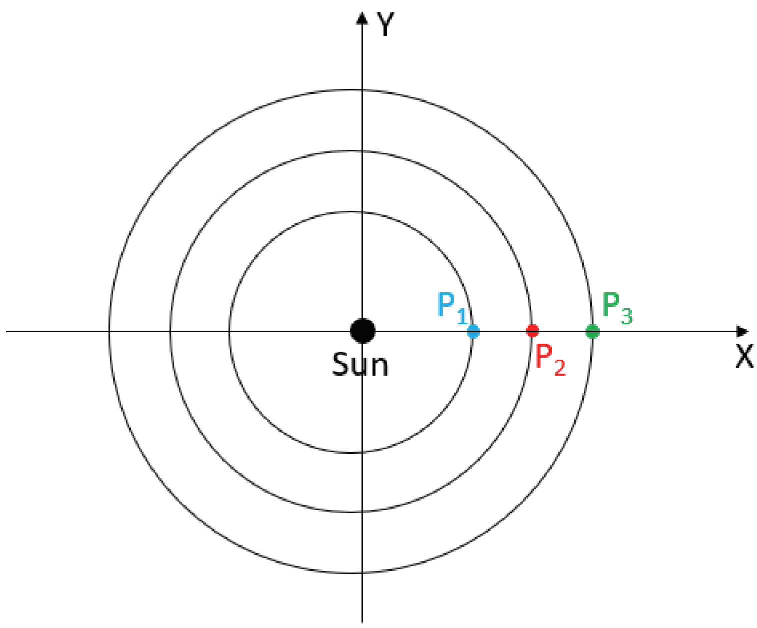


Figure 2. Simplified heliocentric planetary model. All planets ($P_1$, $P_2$, $P_3$) have circular orbits centred at the sun. At time = 0, all planets are located on the x-axis (i.e., they have "zero phase").

As shown in Figure 3, if the orbital radii of observed planet $(P)$ and the Earth $(E)$ be $R_P$ and $R_E$ respectively, then the coordinates of the observed planet *in Earth's frame of reference* are:

$$x_{PE} = x_P - x_E = R_P \cos\theta_P - R_E \cos\theta_E$$
$$y_{PE} = y_P - y_E = R_P \sin\theta_P - R_E \sin\theta_E$$

If the planet $P$ completes one full revolution ($2\pi$ radians) around the sun in time $T_P$, then in time $t$, it would have covered an angle $\theta_P = 2\pi t/T_P$. We say that $T_P$ is the orbital period of the planet $P$. Similarly, if the orbital period of Earth $E$ is $T_E$, then in the same time $t$, it will have covered an angle $\theta_E = 2\pi t/T_E$.

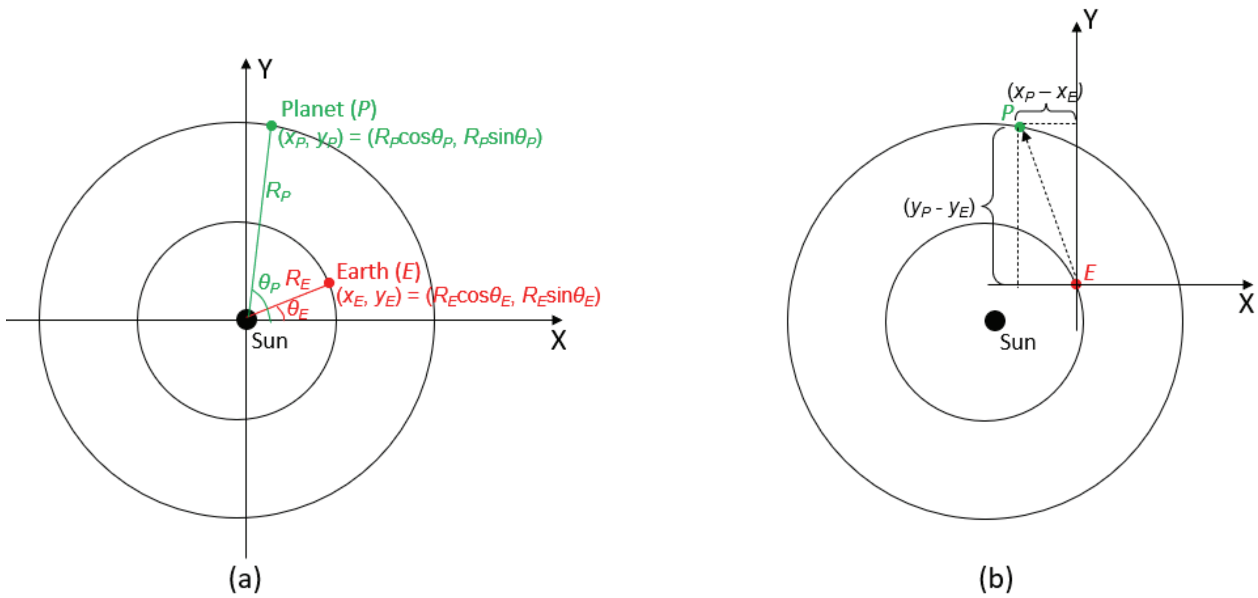


Figure 3. (a) The coordinates of planets $E$ and $P$ with the Sun as the origin. (b) The coordinates of planet P with planet E as the origin.

So, we can rewrite:

$$x_{PE} = x_P - x_E = R_P \cos\left(2\pi t/T_P\right) - R_E \cos\left(2\pi t/T_E\right)$$
$$y_{PE} = y_P - y_E = R_P \sin\left(2\pi t/T_P\right) - R_E \sin\left(2\pi t/T_E\right)$$

Note that after a period $T = \text{LCM}\left(T_P, T_E\right)$, both planets will have completed an integer number of revolutions and will be back in their initial positions on their respective orbits. A plot of $(x_{PE}, y_{PE})$ will therefore, also return to its starting point after this time and then start repeating itself beyond this time.

Finally, we also note that in the above equations, $\theta$ increases with $t$ for all planets. The implication is that all planets orbit the sun in the same direction. This is indeed true for our solar system.

### Mars and Venus

If the orbital radii of Earth, Mars and Venus be respectively $R_E$, $R_M$ and $R_V$ and their orbital periods respectively be $T_E$, $T_M$ and $T_V$, then $R_M \approx 1.5R_E$, $R_V \approx 0.7R_E$, $T_M \approx 1.9T_E$, and $T_V \approx 0.6T_E$ [2, pp. 388–389]. Using these values in the above equations, we plot the orbit of Mars for $\text{LCM}\left(1.9, 1\right) = 19$ years and the orbit of Venus for $\text{LCM}\left(0.6, 1\right) = 3$ years as seen from Earth. Figure 4 shows the results obtained using the Desmos online graphing calculator [3]. The small loops highlighted in the insets represent periods of 'apparent retrograde motion' where the observed planet seems to reverse its 'usual' direction of motion against the backdrop of the distant stars.

### Correlating the Math with the Physical System

We define the planets to be at 'perigee' when they are closest to each other and at 'apogee' when they are furthest from each other. As shown in Figure 5, the 'perigee' distance between the two planets $P$ and $E$ will be $R_P - R_E$ and the apogee distance between them will be $R_P + R_E$.

Indeed, we can confirm from Figure 4 that, the minimum (perigee) distance between the Earth (at origin) and the observed planet (on the curve) is $|R_P - R_E|$ (which is 0.5 for Mars and 0.3 for Venus) and the

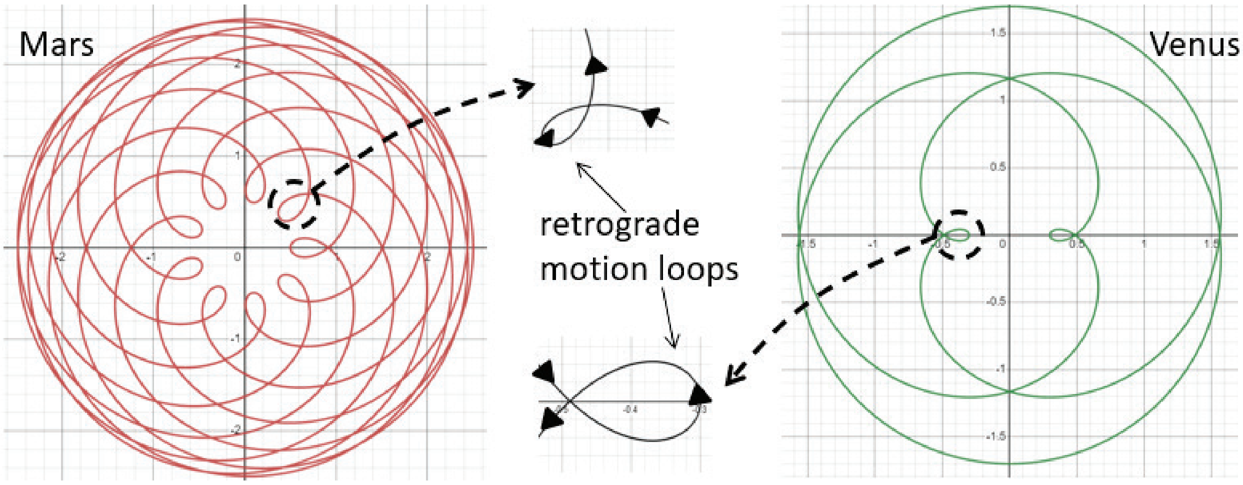


Figure 4. Apparent motion of Mars and Venus observed from the Earth in our model.

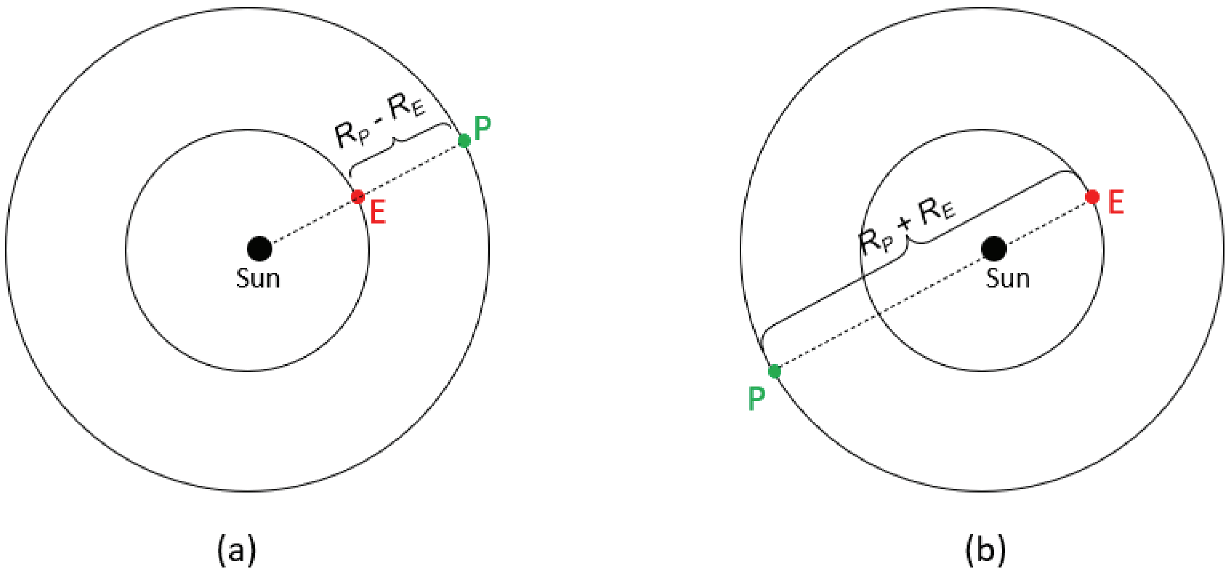


Figure 5. (a) Perigee: when the planets are closest to one another.
(b) Apogee: when the planets are furthest from one another.

maximum (apogee) distance is $R_P + R_E$ (which is 2.5 for Mars and 1.7 for Venus). We now analyze some more features of the graphs obtained in Figure 4.

## Retrograde loops

Let us define $P$ to be an *inner planet* if $R_P < R_E$ and an *outer planet* if $R_P > R_E$. (The usual nomenclature is 'inferior' and 'superior'. Note that though Mars is to be treated as an "outer" planet for this article, it is otherwise considered an inner planet as it is inside the asteroid belt.) At equally spaced times $t_1$, $t_2$ and $t_3$, let the Earth $E$ be at locations $E_1$, $E_2$ and $E_3$ and the observed outer planet $P$ be at locations $P_1$, $P_2$ and $P_3$ respectively, with both $E$ and $P$ orbiting the sun in the same (anticlockwise) direction. Figure 6 shows three possible configurations for the planets' positions at $t_1$, $t_2$ and $t_3$ as case A, case B and case C.

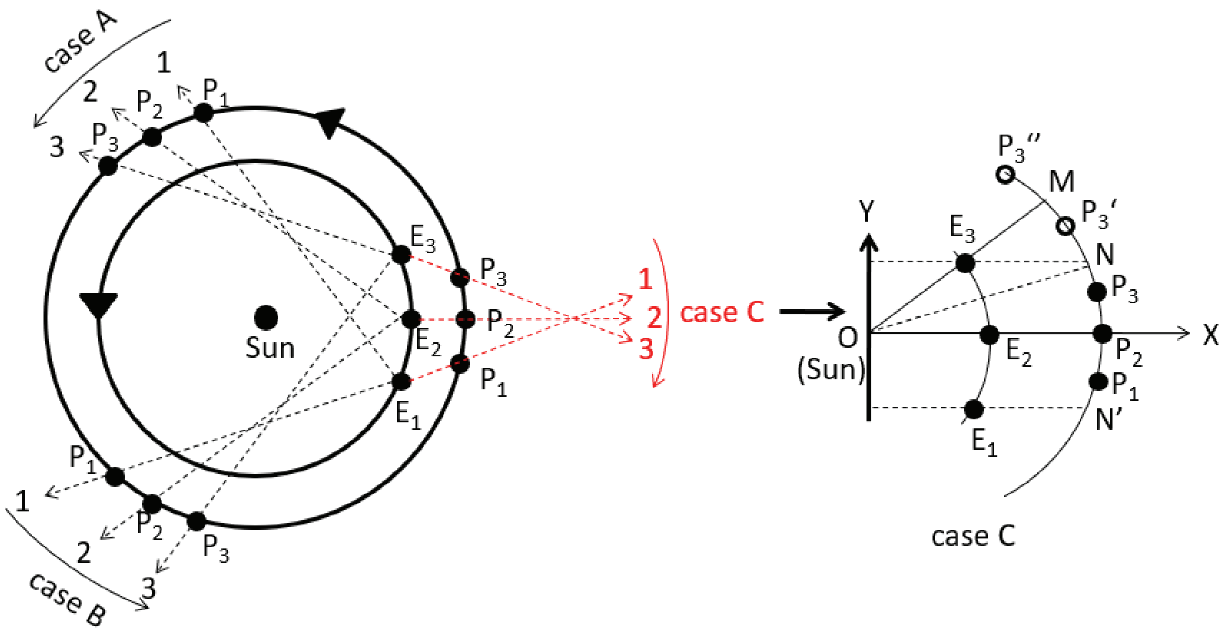


Figure 6. Apparent retrograde motion occurs when planets are positioned as in case C.

In case A and case B, the apparent motion of $P$ as seen from $E$ seems to be anticlockwise. In case C however, $P$ appears to have reversed its motion and moves clockwise when seen from $E$. Such retrograde motion is only possible at the perigee if some additional constraints are met as discussed below.

With reference to the inset for Case C in Figure 6, if the lines $E_3N$ and $E_1N'$ be parallel to the line $OE_2P_2$, then apparent retrograde motion will only occur if $P_3$ lies between $P_2$ and $N$ and by symmetry $P_1$ lies between $P_2$ and $N'$. If $P$ were to move faster so that it reaches position $P_3'$ between $N$ and $M$ at $t_3$, then no retrograde motion would be seen. In particular, if $P$ were to reach position $P_3''$ beyond $M$ at $t_3$, it would mean that the orbital period of $P$ is less than that of the Earth. Thus, for retrograde motion we want the y-coordinate of $P_3$ to be less than that of $N$ or equivalently, $E_3$, i.e., $R_P \sin(2\pi t/T_P) < R_E \sin(2\pi t/T_E)$, where $t = t_3 - t_2$. Since $\sin\theta \approx \theta$ for small $\theta$, we can write the condition for retrograde motion for at least an infinitesimal time for outer planets as:

$$1 < R_P/R_E < T_P/T_E$$

A similar analysis yields the condition for retrograde motion for inner planets as $1 > R_P/R_E > T_P/T_E$. By symmetry, this analysis applies to all perigee locations.

All the planets in the solar system show retrograde motion because they satisfy these conditions as a consequence of Kepler's Law: $T_E^2/R_E^3 = T_P^2/R_P^3$ which in turn is a consequence of Newton's theory of Gravitation [2, pp. 388-389, p. 404].

If $T = LCM(T_P, T_E)$, then the number of retrograde loops formed will depend on how many times the outer planet gets lapped by the inner planet in time $T$. Thus, number of retrograde loops in the plot will be $|T/T_E - T/T_P|$. These loops will be evenly distributed in the 360° angle around the Earth. This explains why in Figure 4, the Mars plot has $|19 - 10| = 9$ retrograde loops at every $360°/9 = 40°$, while the Venus plot has $|3 - 5| = 2$ loops at every $360°/2 = 180°$.

## Initial Phase

We assumed that both the Earth and the observed planet start in their orbits with zero initial phase. In this scenario, as shown in Figure 7, we would expect at least one perigee to occur in the $+x$ direction for outer planets and along the $-x$ direction for inner planets. These have been highlighted for Mars and Venus in Figure 7. Since Venus has its two perigees separated by 180°, its other perigee also ends up on the $x$-axis, along the positive $x$-axis.

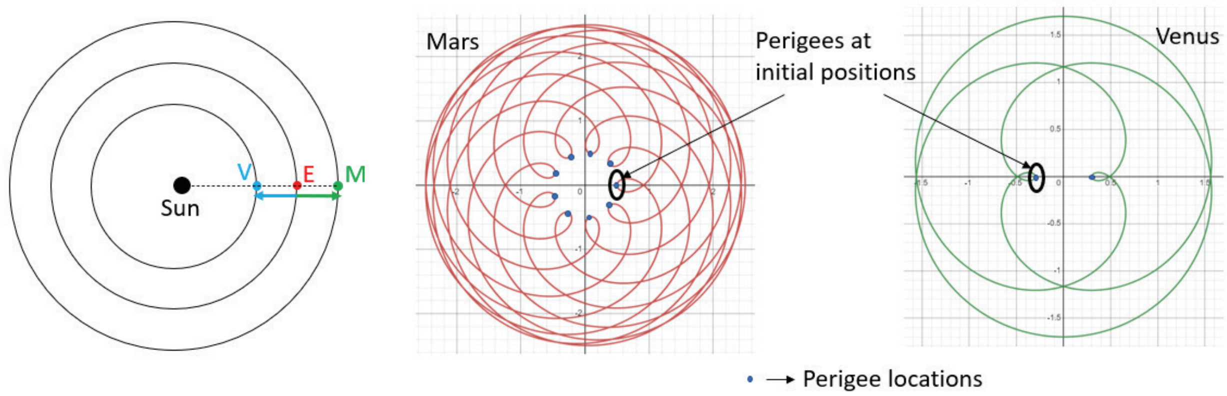


Figure 7. Initial positions are positions of perigee along the $x$-axis when all planets start with zero phase.

Now, let the initial phases of the Earth and the observed planet be $\varnothing_E$ and $\varnothing_P$ respectively, then the first perigee location will be reached when $\frac{2\pi t}{T_P} + \varnothing_P = \frac{2\pi t}{T_E} + \varnothing_E$. For example, if Venus starts at phase 0 and Earth starts at phase 90°, then a perigee will occur when $\frac{2\pi t}{0.6} = 2\pi t + \pi/2$, i.e., $t = 3/8$ and the phase for both planets at this time is 225°. The whole orbit of Venus as seen from Earth in this situation would then appear to be rotated w.r.t. Figure 4 by this angle. This is borne out by Figure 8.

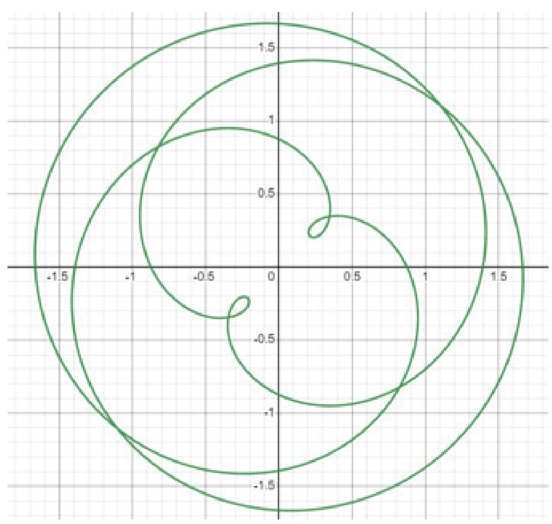


Figure 8. The graph is rotated in accordance with the relative initial phase of the planets.

## Hypothetical planets

Figure 9 shows what would have happened in the hypothetical scenario where Mars and Venus orbit the sun in a direction *opposite* to that of the Earth.

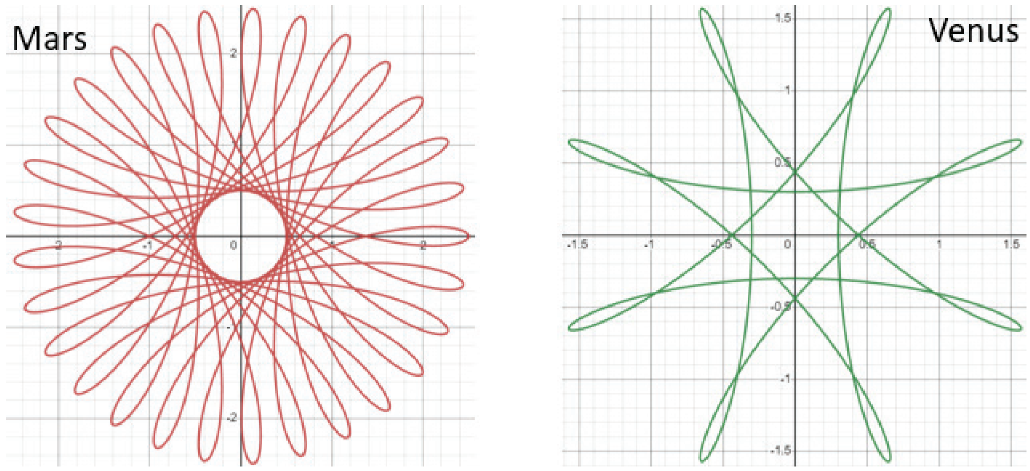


Figure 9. Mars and Venus plots from Earth if orbiting in opposite direction to Earth.

The conditions for the formation of the retrograde motion loops remain the same, except that they are now formed at the apogee. This causes the retrograde loops to face outward. The number of times the planets are at the apogee in time $T = LCM(T_E, T_P)$ in this case, is $(T/T_E + T/T_P)$. Hence, Mars now shows $19 + 10 = 29$ loops of retrograde motion while Venus shows $3 + 5 = 8$ such loops.

### The KVPY Planets

Let us now turn back to the function given in the KVPY question. Ignoring the modulus operation (we will come back to this later), we can rewrite [4]:

$$x(\theta) = \cos 4\theta \cos\theta = 0.5(\cos 5\theta + \cos 3\theta) = 0.5(\cos 5\theta - \cos(\pi - 3\theta))$$
$$y(\theta) = \cos 4\theta \sin\theta = 0.5(\sin 5\theta - \sin 3\theta) = 0.5(\sin 5\theta - \sin(\pi - 3\theta))$$

Relating this to our model of apparent planetary motion, we can now deduce:

1. The orbital radii of the two "planets" are same, therefore, the nearest distance between them will be zero. All curves satisfy this since they pass through $(0, 0)$.
2. While one planet's motion is determined by $+\theta$, the other's changes as $-\theta$, i.e., these planets are orbiting in opposite directions. Therefore, the retrograde motion loops must face outward. This is also satisfied by all the curves.
3. The number of retrograde motion loops should be $(5 + 3) = 8$. This eliminates options (B) and (D).
4. Since the reference planet has initial phase $\pi$ while the observed planet has initial phase 0, the starting position must be an apogee position *along the x-axis*. This does not occur in (C). Thus, the answer must be (A).

Can you now work out what the equations could be for the remaining curves?

### What about the Modulus?

With the help of the DESMOS calculator, we plotted the graph for the functions with and without the modulus operation for increasing ranges of $\theta$. The results are shown in Figure 10.

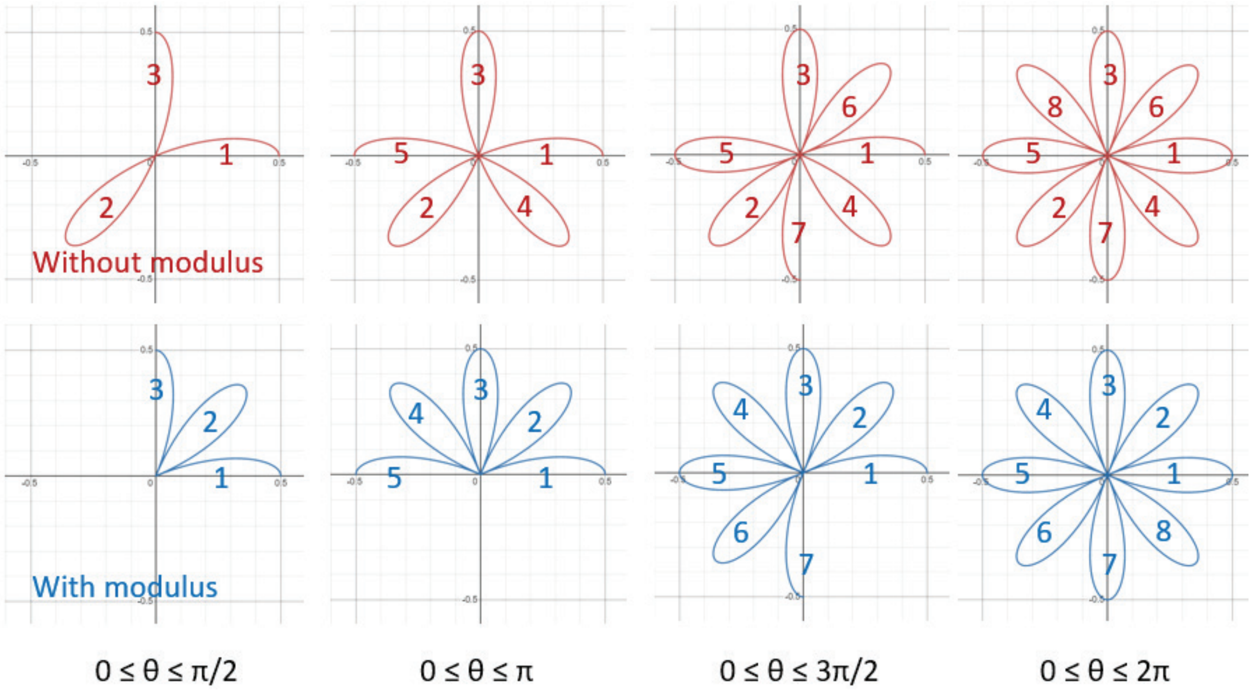


Figure 10. Difference caused by modulus operation.

The sequence in which the retrograde loops are produced is different in the two cases. Once $\theta$ has covered its full period of $2\pi$ though, the graphs are indistinguishable. We wonder what physical system could correspond to the equations with the modulus?

## Conclusions

Our analysis of planetary motion helped us appreciate the KVPY question discussed in this article from a completely different perspective. We wonder if other equations and graphs that we come across in puzzle books and examinations are also related to some everyday physical phenomena. We believe an awareness of such connections will hugely enrich our learning of both, the physical phenomena and its underlying mathematics. In that context, here are a few more points to ponder as extensions of the discussion in this article (ignore the modulus operation to begin with):

1. We said that the graph loops back to its starting point and then repeats after time $T = \text{LCM}\,(T_P, T_E)$. What is $T$ if $T_P/T_E$ is not rational? What happens to the graph in this case?
2. What if both planets start with the same but non-zero initial phase?
3. Given a function, can we predict the sequence in which the retrograde loops will be generated as $\theta$ increases (see Figure 10)?
4. Can we predict how many "layers" of intersections the curve will have and the angles along which these intersections will lie (see Figure 11)?
5. What if $1 < T_P/T_E < R_P/R_E$ for outer planets? Does the curve have concavities or is it fully convex or does it depend on exactly how much $R_P/R_E$ is greater than $T_P/T_E$?

**Trivia**: The *Geometric Chuck* is a mechanical instrument that generates the types of curves discussed in this article [5]. Such curves are equivalent to the ancient (geocentric) epicycle model of the solar system which, understandably, had good success in explaining the retrograde motion of planets [6].

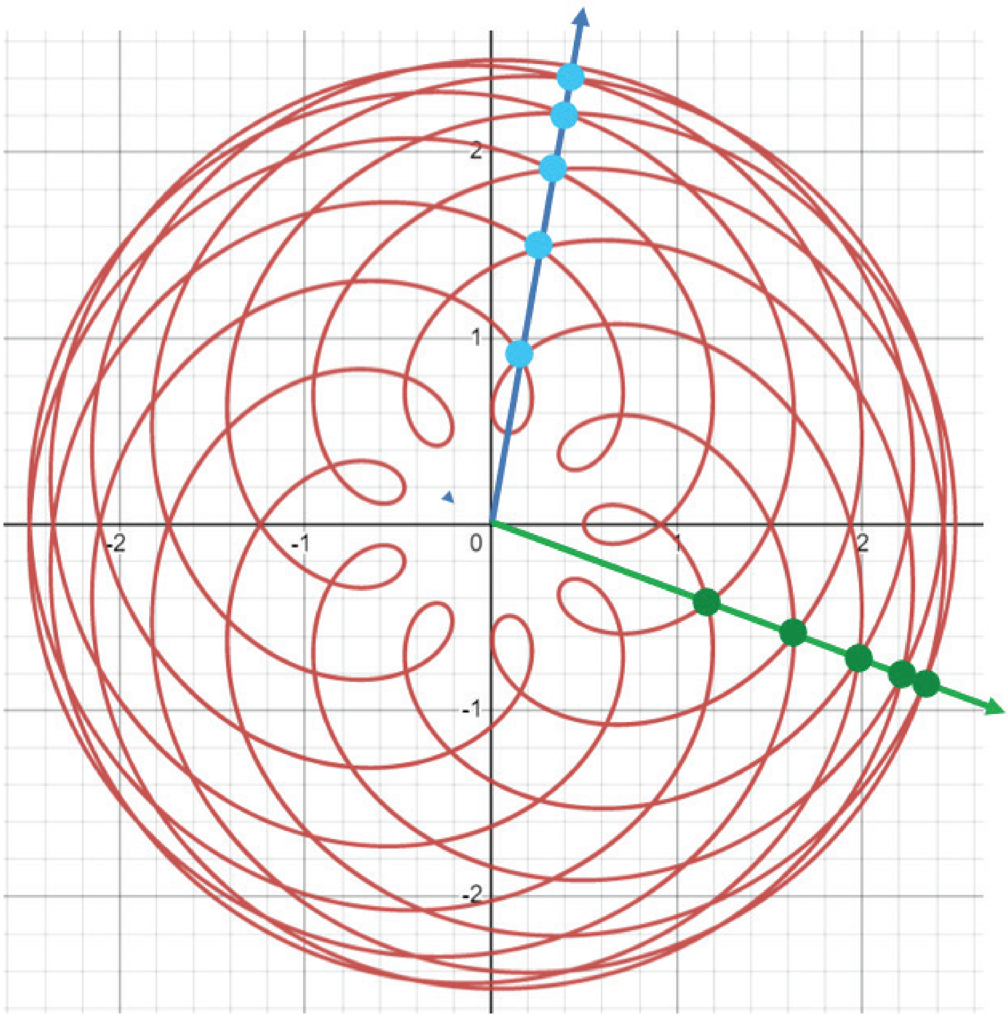


Figure 11. The graph for Mars intersects itself in 5 "layers" (marked by the 5 dots in blue and green colours). The intersections exactly align along radial lines of two types: (a) going through the centre of the retrograde loop and (b) going exactly between two neighbouring retrograde loops.

**Appendix: Desmos graphing calculator**

The Desmos online graphing calculator can be accessed at: https://www.desmos.com/calculator. The user interface is quite intuitive. One can either directly type the equation to be plotted in the box provided or use the inbuilt keyboard option. Settings are available to alter the appearance of the grid and of the plotted curve. Extensive documentation and a broad compilation of example graphs can also be accessed from the tool itself (see Figure A1).

In this article we have used the feature for plotting *parametric* curves. One can refer to the 'Parametric: Introduction' example in the Desmos example list to get started. The parametric form allows us to express the $x$ and $y$ coordinates as a function of the parameter $t$, i.e., $x = f(t)$ and $y = g(t)$. The coordinates of the curve may then be entered as $(f(t), g(t))$ in the box provided for the input equation. For example, the straight line equation $y = 3x$ can be plotted in parametric form as $(t, 3t)$ as shown in Figure A2. The default Desmos range of the parameter $t$ is $0 \leq t \leq 1$, which can be changed as per our needs.

For this article, we need the coordinates of, for example, Mars ($M$) with the Earth ($E$) as the origin. Since both planets are assumed to start with the same (zero) phase, be in circular orbits with the sun at the center

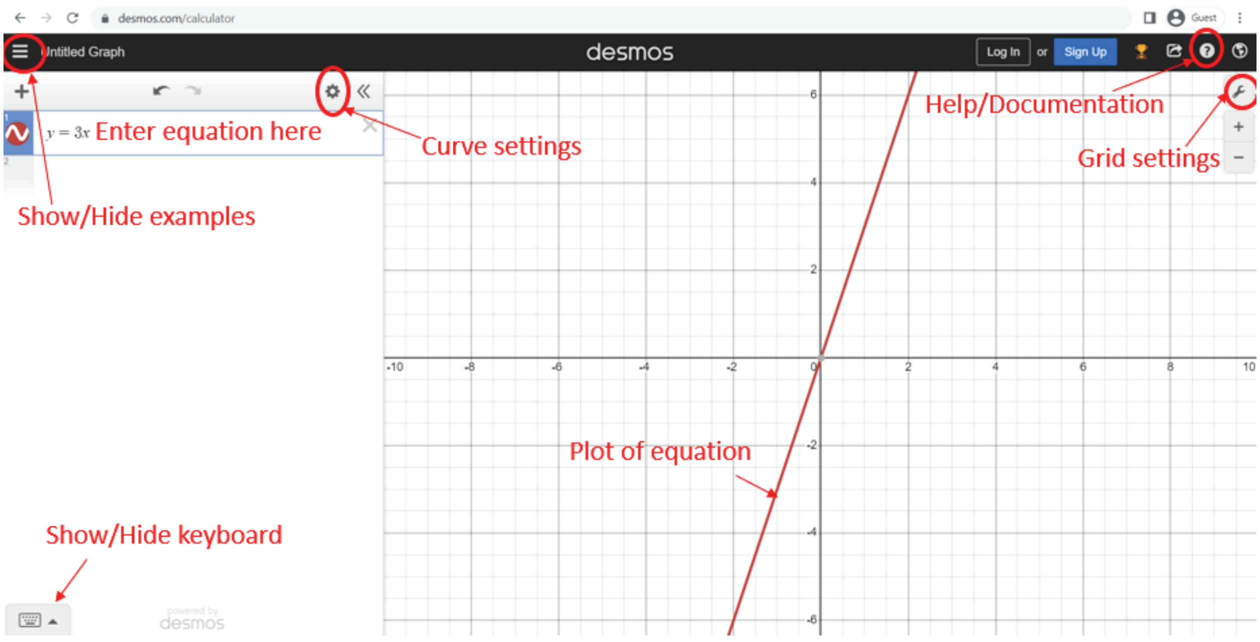


Figure A1. Desmos user interface.

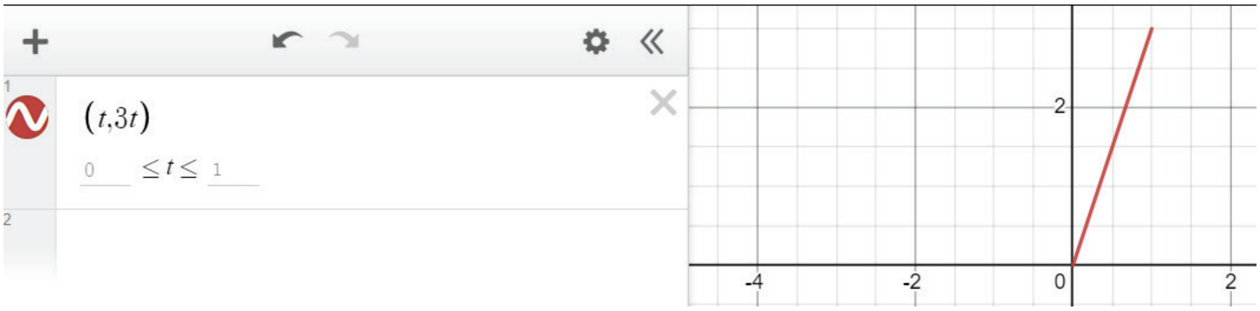


Figure A2. Parametric form of equation $y = 3x$ expressed as $(t, 3t)$ with $0 \leq t \leq 1$.

and revolve in the same direction, their coordinates, with the sun as the origin, at any time $t$, can be represented as shown in Figure A3.

With $T_M$ and $T_E$ being respectively the orbital periods of Mars and Earth, both $\theta_M$ and $\theta_E$ can be expressed in terms of time $t$. The coordinates of Mars w.r.t. Earth at any time $t$ can then be written as:

$$x_{ME} = x_M - x_E = R_M \cos(2\pi t/T_M) - R_E \cos(2\pi t/T_E)$$
$$y_{ME} = y_M - y_E = R_M \sin(2\pi t/T_M) - R_E \sin(2\pi t/T_E)$$

Further, knowing that $R_M \approx 1.5R_E$, we can put $R_E = 1$ and $R_M = 1.5$. Similarly, knowing that $T_M \approx 1.9T_E$, we can put $T_E = 1$ and $T_M = 1.9$. Thus, we can write the coordinates for Mars in Earth's frame of reference in Desmos as:

$$(1.5\cos(2\pi t/1.9) - \cos(2\pi t), 1.5\sin(2\pi t/1.9) - \sin(2\pi t))$$

This generates the graph shown in Figure A4 since the range used for $t$ is still set to the default. If we extend this range from 0 to LCM $(T_M, T_E) = 19$, we will get the curve for Mars as shown in Figure 4.

Similarly, the curve for Venus ($V$) shown in Figure 4 is generated using the coordinates:

$$(0.7\cos(2\pi t/0.6) - \cos(2\pi t), 0.7\sin(2\pi t/0.6) - \sin(2\pi t))$$

where t ranges from 0 to LCM $(T_V, T_E) = 3$.

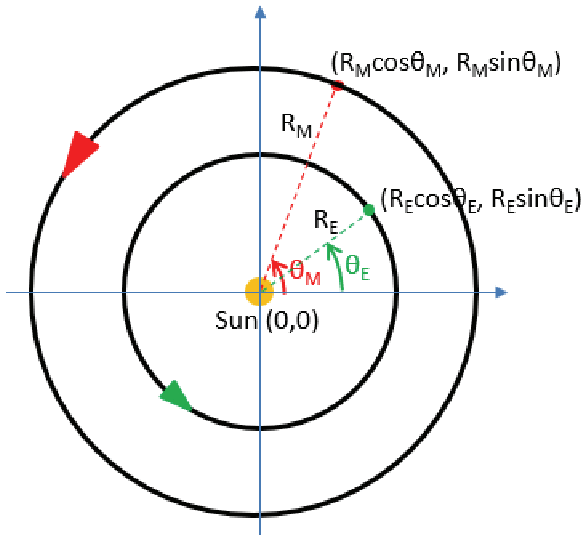


Figure A3. Coordinates of planets $E$ and $M$ with the Sun assumed to be at origin.

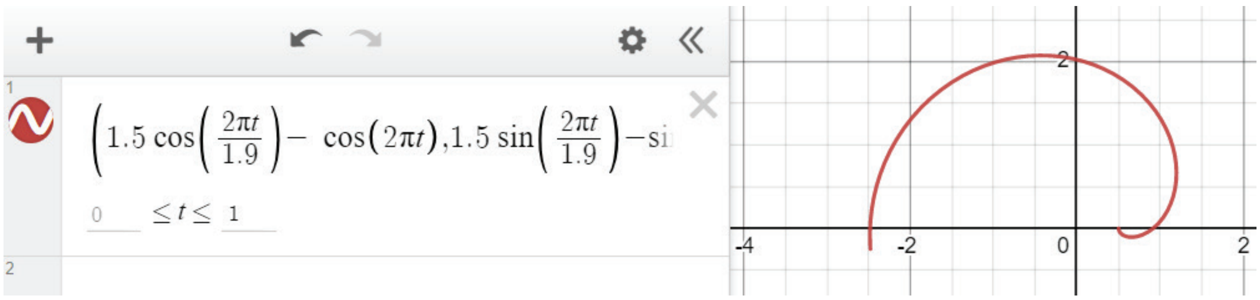


Figure A4. Plot of Mars w.r.t. Earth with default range for t.

To generate the curve for Figure 8, it was assumed that Earth had an initial phase of $\pi/2$. The coordinates of Venus w.r.t. Earth are then given by:

$$(0.7\cos(2\pi t/0.6) - \cos(2\pi t + \pi/2), 0.7\sin(2\pi t/0.6) - \sin(2\pi t + \pi/2))$$

For Figure 9, we have to assume that Mars and Venus revolve opposite to Earth's direction, hence their $\theta$ changes as $-t$ while Earth's changes with $t$. The coordinates for Mars and Venus w.r.t. Earth then respectively become:

$$(1.5\cos(2\pi(-t)/1.9) - \cos(2\pi t), 1.5\sin(2\pi(-t)/1.9) - \sin(2\pi t))$$
$$(0.7\cos(2\pi(-t)/0.6) - \cos(2\pi t), 0.7\sin(2\pi(-t)/0.6) - \sin(2\pi t))$$

The coordinates and the parameter ranges used to generate Figure 10 have already been provided in the main article (to enter them in Desmos, use $t$ instead of $\theta$).

## References


1. Kishore Vaigyanik Protsahan Yojana SX/SB (2017), *Question 17*, pp. 12–13. Available at: http://kvpy.iisc.ac.in/main/resources/2017/SB_05-NOV-17.pdf
2. Newton, I. (1687). *The Mathematical Principles of Natural Philosophy*. (Motte, A., trans.) New York, NY: Daniel Adee. Available at: https://ia802706.us.archive.org/0/items/newtonspmathema00newtrich/newtonspmathema00newtrich.pdf
3. Desmos. Available at: https://www.desmos.com/calculator

4. Byju's. *Trigonometric Formulas: Sum to Product Identities*. Available at: https://byjus.com/maths/trigonometry-formulas/#sum-to-product-identities

5. Bazley, T. S. (1875). *Index to the Geometric Chuck.* London, UK: Waterlow and sons. Available at: https://archive.org/details/indextogeometri00bazlgoog/mode/2up

6. *Trochoidal Curves*, The Penny Cyclopaedia (1843), Volume 25, London, UK: Charles Knight, pp. 282–290. Available at: https://archive.org/details/pennycyclopdias01longgoog/mode/2up

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