Student Corner

Deriving an Equation for the Sun's Path

AMAN MAKHIJA

Watching the sun rise and set from my balcony each day, I wondered: how easy is it to derive a mathematical formula to describe the path of the sun over the course of the day?

This path varies depending on the observer's latitude and the time of year.

To solve this problem, we must account for the fact that the Earth has two rotational movements: it rotates around its own axis once every 24 hours, and it also revolves around the sun once every year. Both these are relevant to this problem. Luckily for us, they are independent of each other, and we can tackle them one at a time.

Understanding this problem involves three key concepts.

Concept 1: It is helpful to pretend the universe rotates around the Earth's axis, instead of the other way round

The first motion we consider is the Earth's rotation about its axis over a 24-hour period. Since we are interested in the path traced by a celestial body (the sun) from the Earth's perspective, we invoke a clever trick: we pretend the Earth is stationary and consider the universe to be rotating about the Earth's axis in the opposite direction.

From the point of view of an observer, all celestial bodies will appear to trace circles about a fixed point in the sky which lies on the Earth's axis of rotation. We just need a way to identify this point from a given location on Earth.

Keywords: Solar declination, midnight sun, polar night, vector geometry, mathematization, astronomy, vectors, trigonometry, coordinate geometry

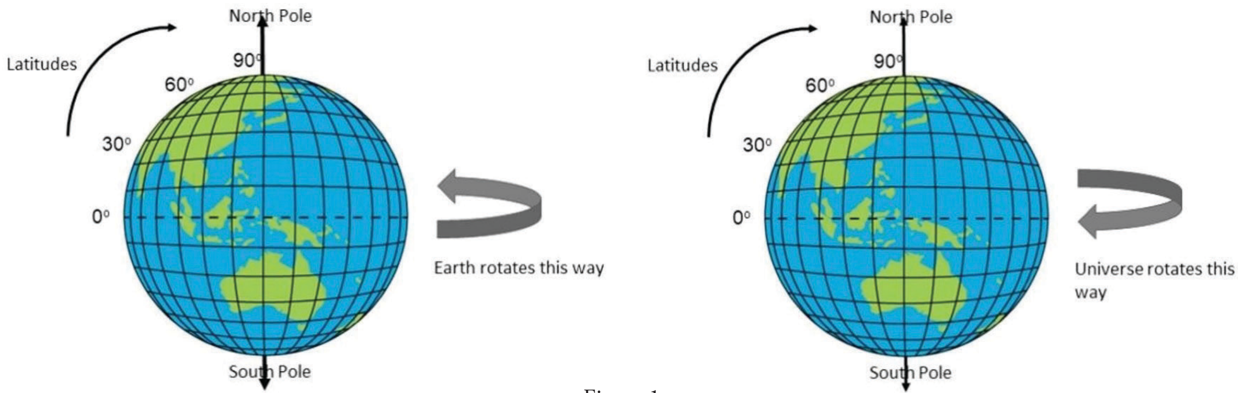

Figure 1

In the proof that follows (Figure 2), it can be seen that the position of this fixed point in the sky depends on the latitude of the observer. In the Northern hemisphere, the point along the axis (known as the North celestial pole or Celestial North) is at an angle of the observer's latitude above his/her northern horizon. This point is marked by the Pole Star (Polaris) in the night sky[1]. In the Southern hemisphere, the fixed point is known as the South Celestial Pole and is elevated at an angle of the observer's latitude above the southern horizon. For our purposes, we will focus on an observer in the Northern hemisphere for convenience.

Proof: Consider a point L on the earth at Latitude ϕ. **OE** is a vector from the centre of the earth to the equator (intersecting at the longitude of point L) and **ON** is the vector pointing from the centre of the Earth to the North Pole (i.e., the Celestial North direction). The local northern horizon is a tangent vector along the earth's surface Let us assume the tangent intersects the vector pointing along the earth's axis at point T.

In Figure 2B, $\angle TOL = 90 - \phi$, because **OT** and **OE** are perpendicular to each other. Vector **LT** is tangent to the sphere and therefore perpendicular to the radius.

$\therefore$ OLT is a right triangle.

$\therefore \angle OTL = \phi = \angle TLN'$ (interior alternate angles).

In other words, at any given point an observer will see Celestial North at an angle equal to the latitude ϕ above his/her local North, as shown in Figure 2.

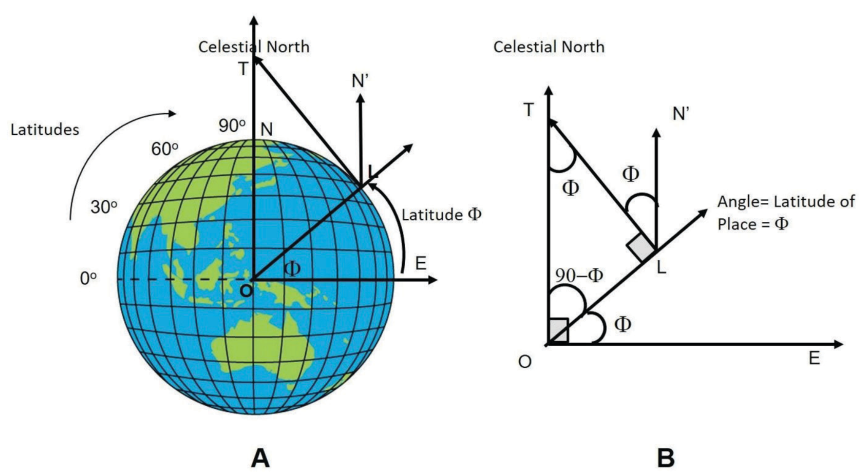

Figure 2. (A) and (B) illustrate why celestial north is at an angle ϕ = the latitude of a place above the local Northern horizon.

[1] Ancient seafarers used the pole star to navigate because its position in the sky is fixed through the day and year, because it lies on the earth's axis.

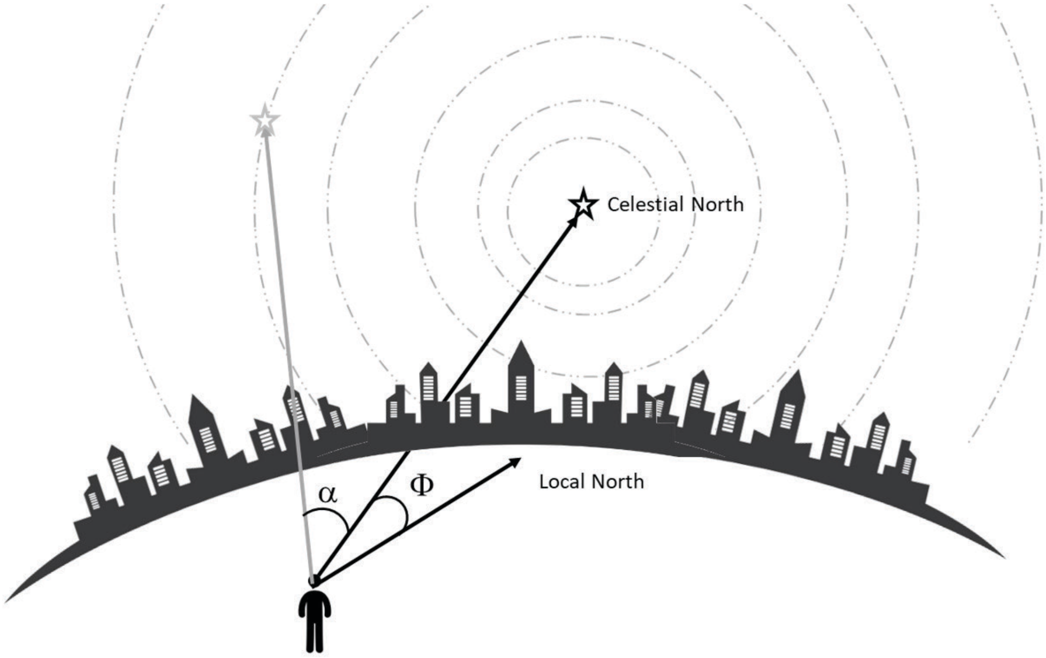

Figure 3. Celestial bodies trace circles around the North Pole.

Concept 2: The size of the circle traced by a celestial body depends on the North Pole Separation Angle of the object in question on that particular day.

Because the Earth's axis is fixed, celestial bodies trace circular paths about the axis of rotation. But what is the exact circular path a given celestial body will trace? This is determined by one specific angle called the "North Pole Separation"[2] of the celestial body (Figure 3).

The North Pole Separation is the angle (α) between the Earth's axis (**ON**) and the vector pointing in the direction of the celestial body. A celestial body with a small α (say 10 degrees), will trace a small circle around celestial north in the northern sky, while a celestial body with a large α (say 60 degrees), will trace a bigger circle around celestial north each day.

Concept 3: For the sun, the North Pole Separation angle (α) changes throughout the year.

For most celestial bodies the North Pole Separation, and hence the paths traced are fixed through the year. The Sun is unique, because its position relative to the Earth changes throughout the year as the Earth revolves.

This revolution and the Earth's axial tilt causes the North Pole separation angle of the sun to change over the course of the year, causing the daily path of the Sun to change.

So, our goal is to solve for α in terms of the time of year.

Let us start by defining our coordinate system, as follows:

- The centre of the Earth is at the point (0,0,0), which we call O.
- X-axis: Axis lying on the plane of revolution and the plane containing the Earth's rotational axis and the Z-axis. It's the direction of the sun on the June solstice.
- Y-axis is thus defined to be perpendicular to both the X and Z axes.
- Z- axis: Axis perpendicular to the plane of revolution and passing through the centre of the Earth.

[2] Astronomy texts will generally refer to the complementary angle to the North Pole separation, known as declination. But for the purposes of this problem, using the North Pole separation is more convenient.

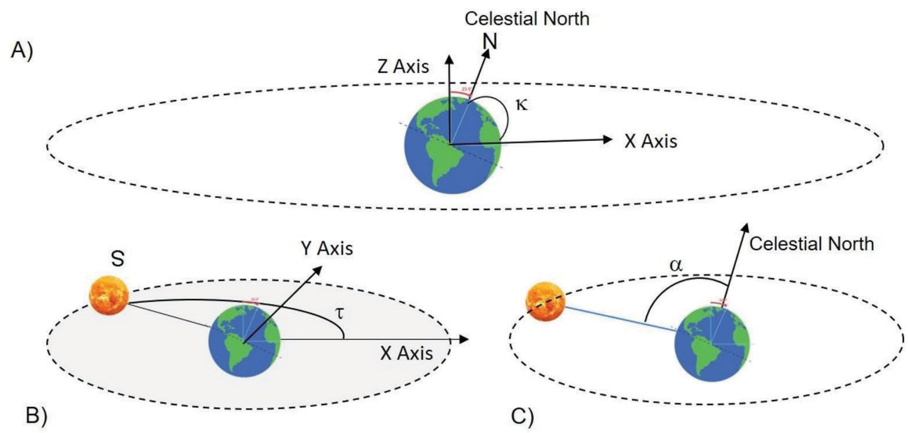

Figure 4. The coordinate system

We assume that the sun moves anti-clockwise along the unit circle centred at O, completing one complete cycle every year. The sun is along the X-axis on the June solstice. The XY plane is the plane of the sun's revolution.

- The unit vector in the direction of the Earth's rotational axis (in the Northern direction) is N. The angle between the Earth's rotational axis and the plane of revolution is a constant κ = 66. 5 degrees (Figure 4A).
- τ is the angle traced out by the Sun in its orbit since June 21 in radians (Figure 4B). For example, 37 days after summer solstice, τ = (37/365) × 360 degrees.
- The unit vector from the centre of the Earth in the direction of the sun is called S.
- α is the angle between S and N or the north pole separation of the sun (Figure 4C).

As the circular path traced by the sun depends on the North Pole Separation (α), the mathematical question is, what is the exact formula for α at a given time of the year (represented by τ)?

Figure 5 shows that α is the smallest on the June solstice (it is 66.5 degrees on this day) when the northern hemisphere is tilted towards the sun, and largest on the December solstice when the northern hemisphere is tilted away from the sun (it is 113.5 degrees). On the equinoxes, α is 90 degrees.

We can find α in terms of τ by using vector analysis.

Using the coordinate system defined, the position vector of the sun on any given day is:

$$S = [cos(\tau), sin(\tau), 0]$$

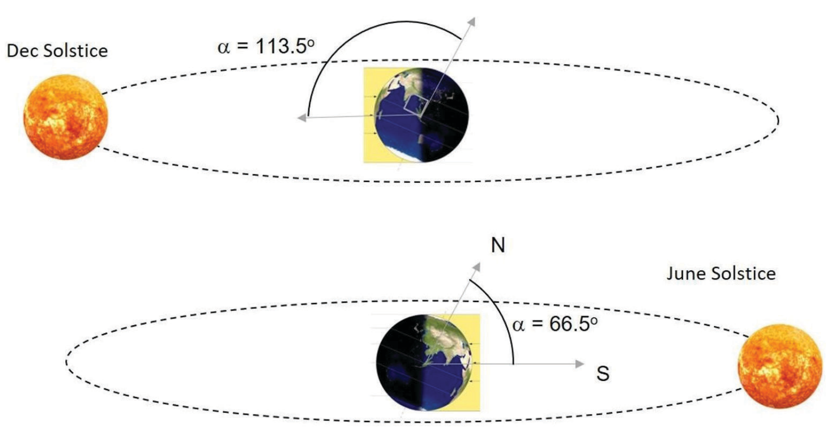

Figure 5. The North Pole separation angle α, varies through the year.

Since the earth is fixed in our convention system, N is constant with time, with the y coordinate as 0 by definition. Its x coordinate is cos (κ) and the z coordinate is sin(κ), so the position vector N is

$$N = [cos(\kappa), 0, sin(\kappa)]$$

In order to find the cosine of the angle between S and N, we take the dot product of the two vectors. Since the y and z component products are clearly 0, we end up with only x-component product:

$$cos(\alpha) = cos(\tau) * cos(\kappa)$$

We have now found a formula for α, the angle of separation, in terms of time of year (τ).

Discussion: What does it mean? The overhead sun.

From Figure 3, we can draw some conclusions about specific cases of the position of the celestial object.

For example, if $\alpha + \phi = 90^o$ then the object will pass overhead.

For the sun, we can use the formula $cos(\alpha) = cos(\tau) * cos(\kappa)$ and set $\alpha = 90^o - \phi$ to get

$$cos(\tau) = sin(\phi)/cos(\kappa)$$

We can then solve for τ to find the day of the year when the sun passes overhead between the tropics.

For example, let us find the dates on which the sun passes overhead in Bengaluru, Karnataka, India (at a latitude of $\phi = 12.8°$ N). We get $cos(\tau) = sin12.8°/cos66.5°$ which gives $\tau = 56°$ or $\tau = 304°$.

Since τ is the proportion of the year completed since June Solstice (June 21), we get that the sun passes overhead in Bengaluru $\tau/360 * 365$ days after June 21.

This corresponds to 17 August and 25 April, for the two values of τ [3] (slight variation in the time of summer solstice may cause it to vary – but by at most a day!).

The same formula can be used to explain the phenomena of the "midnight sun" and "polar night" in far-North latitudes ($\phi > 66.5^o$).

From Figure 3, we see that the case with $\alpha < \phi$ in the Northern hemisphere for a particular celestial body would mean it would never set below the local horizon over the 24-hour period.

In the case of the sun, this occurs near the June Solstice (Summer) in far-North locations; this is referred to as the "midnight sun" because the sun is above the horizon, throughout the day.

Similarly, the case where $\alpha + \phi > 180$ degrees occurs near the December Solstice (Winter); this is referred to as the "polar night", because the sun never rises over the course of the 24-hour period.

[3] https://bangaloremirror.indiatimes.com/bangalore/others/its-a-zero-shadow-day-in-bengaluru-at-noon-today-the-sun-will-be-exactly-overhead/articleshow/63887232.cms

AMAN S. MAKHIJA is a grade 12 student living in Bangalore. An aspiring mathematician, he loves geometry and linear algebra. He is also an avid stargazer and music lover. He can be contacted at amanmakhija1010@gmail.com