Student Corner

A Number Theory Problem from Hungarian Math Olympiad

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In this article we discuss a solution to a Number Theory Problem, which was given in the 80th Eötvös-Kürschák Competition, 1980 [1].

Problem. Let $n > 1$ be an odd integer. Prove that a necessary and sufficient condition for the existence of positive integers x and y satisfying

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y}$$

is that n has a prime divisor of the form $4k - 1$.

Solution. We first prove that the given condition is sufficient. (This is the easy part.)

Assume that n has a prime divisor of the form $4k - 1$. Then there exists a positive integer m such that

$$n = (4k - 1)m.$$

We now have:

$$\frac{4}{n} = \frac{4}{(4k-1)m} = \frac{(4k-1)+1}{k(4k-1)m} = \frac{1}{km} + \frac{1}{k(4k-1)m}.$$

Thus we can take $x = km$, $y = k(4k - 1)m$; they are positive integers satisfying the given equation.

Next we prove that the condition is necessary.

Keywords: Number theory, greatest common divisor, prime divisor, Eötvös-Kürschák Competition

Assume that for the odd integer $n > 1$ there exist positive integers x and y such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y}.$$

We shall prove that n has a prime divisor of the form $4k - 1$.

Let $\gcd(x, y) = d$. Then there exist positive integers u and v such that $x = du$ and $y = dv$, and $\gcd(u, v) = 1$ (i.e., u, v are coprime).

Then from the equality

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y}$$

we get

$$n = \frac{4xy}{x+y} = \frac{4d(uv)}{u+v}. \tag{1}$$

We now prove that $\gcd(uv, u+v) = 1$.

Let p be a prime factor of uv and $u + v$. Since p is a divisor of uv, it must divide either u or v. Let us suppose that p is a divisor of u. Since p is a divisor of u as well as $u + v$, it must be that p is a divisor of v too. But this contradicts our supposition that u and v are coprime. Likewise if p is a divisor of v. We conclude that

$$\gcd(uv, u+v) = 1. \tag{2}$$

From (1) we have,

$$n(u+v) = 4d(uv). \tag{3}$$

Therefore, 4 is a divisor of $n(u + v)$. Since n is odd we deduce that

$$4 \text{ is a divisor of } u+v. \tag{4}$$

From (4) it follows that u, v are either both odd or both even. They cannot both be even as u, v are coprime. Hence u, v are both odd.

Further, since 4 is a divisor of $u + v$, it follows that one of u, v is of the form $4a - 1$ while the other is of the form $4b + 1$. From this we deduce that

$$uv \equiv -1 (\bmod 4). \tag{5}$$

This implies that uv has a prime divisor p of the form $4k - 1$. It follows from (3) that p divides $n(u + v)$.

But since $\gcd(uv, u + v) = 1$, it follows that p does not divide $u + v$. From this it follows that p divides n.

We have thus shown that n has a prime divisor p of the form $4k - 1$, as required. $\square$

Another argument to prove that the given condition is necessary (contributed by the second author). For this, we shall assume that all the prime factors of n are of the form $1 (\bmod 4)$. From this we proceed to derive a contradiction.

Let $d = \gcd(x, y)$ so that $x = du$ and $y = dv$ where $\gcd(u, v) = 1$.

We have:

$$4xy = n(x+y), \quad \therefore\ 4xy = nd(u+v), \quad \therefore\ 4\left(\frac{x}{d}\right)y = n(u+v), \tag{6}$$

so $4uy = n(u+v)$. As $\gcd(n,4)=1$, we get

$$u+v \equiv 0 (\text{mod} 4). \tag{7}$$

On the other hand, from $4uy = n(u+v)$, it follows that u is a factor of $n(u+v)$.

Since $\gcd(u, u+v) = 1$, it follows that u is a factor of n.

Similarly, we conclude that v is a factor of n. So both u, v are factors of n.

By assumption, therefore, $u \equiv 1 (\text{mod} 4)$ and $v \equiv 1 (\text{mod} 4)$. From these we get

$$u+v \equiv 2 (\text{mod} 4). \tag{8}$$

This contradicts (7). We have found the desired contradiction.

It follows that n has a prime divisor of the form $4k-1$, as required . $\square$

References

1. *Eötvös-Kürschák Competitions* (Mathematical and Physical Society), compiled by Ercole Suppa.
http://www.batmath.it/matematica/raccolte_es/ek_competitions/ek_competitions.pdf

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