Student Corner

A Problem from the 2019 Zonal Informatics Olympiad

ADHISH KANCHARLA

Introduction

In this article, I describe an application of the powerful technique of *recursion* to an Informatics Olympiad problem. You can find an introduction to this concept in the July 2014 issue of *At Right Angles*[a].

Problem statement

A sequence of positive integers $[a_1, a_2, a_3, \ldots, a_n]$ is called a Special Sequence if a_1 divides a_2, a_2 divides a_3, and so on until a_{n-1} divides a_n, and if all the elements are distinct.

For example, $[2, 4, 8, 32]$ is a Special Sequence.

But $[4, 2, 8]$ is not, because 4 does not divide 2.

Similarly, $[2, 4, 4, 8]$ is also not Special, because the elements are not distinct.

You need to find the number of Special Sequences such that all the elements of the sequence are from the set $\{1, 2, \ldots, K\}$.

Suppose $K = 3$. The Special Sequences possible are $[1], [2], [3], [1, 2], [1, 3]$. So, the answer would be 5.

Find the answer for the following values of K:

a) $K = 15$

b) $K = 19$

c) $K = 22$.

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a See "Self-Similarity" by Punya Mishra & Gaurav Bhatnagar, available at 5_Self-Similarity.pdf (azimpremjiuniversity.edu.in)

Keywords: Special Sequence, recursion, investigation, constraints, problem-solving.

Key Observation

Since the elements of the Special Sequence must be distinct and each A_i should divide A_{i+1} for all i in the range $[0, n-1]$, we can see that the elements of the Special Sequence must be strictly increasing.

Thus, the largest element can appear only once and must necessarily appear at the nth position.

Hence the total number of Special Sequences can be obtained by summing up the number of Special Sequences for each possible value of the largest element.

The idea of recursion

A function that is defined in terms of itself is called a *recursive function*. A well-known example is the *Fibonacci sequence* which is defined recursively as $F_n = F_{n-1} + F_{n-2}$ with base cases $F_1 = 1$ and $F_2 = 1$.

The key idea in the solution is to use recursion to compute the number of Special Sequences with a fixed largest element.

Solution

Let f_i be the number of Special Sequences such that the largest element of the sequence is i.

Notice that the final answer would be $\sum_{i=1}^{K} f_i$ because each case is mutually exlusive and there are K possible values of i.

First let us do the easy step of establishing the base case: $f_1 = 1$ because the only possible sequence is [1].

Now we need to find a recurrence relation that helps us find f_i in terms of f_j for $j \neq i$. Here we recall the definition of f. We have fixed the largest and therefore last element of the sequence to be i. Hence, we have a number of options for the other smaller elements of the sequence that come before this last element:

Case 1: There are no other elements in this sequence $\Longrightarrow$ the number of Special Sequences $= 1$.

Case 2: There is at least 1 other element in this sequence.

Let us consider all possible values of j, the next largest element in this sequence. We know that this element must come just before the last element and hence it must be a divisor of the last element.

Further, once we fix this second last element as j we see that we have come across a similar subproblem of the same nature and f_j will give us the number of Special Subsequences with the largest element, j.

Thus, simply adding all f_j for $j < i$ such that j divides i will give us the value of f_i.

A way to look at case 1 and case 2 together

Once we fix the largest element of the sequence (i), we just need to add a Special Subsequence to form its beginning. This subsequence can be empty (case 1) but if it is non empty, it must satisfy the condition that j (the largest element of the subsequence being added to the beginning) is a divisor of i (the largest element of the complete sequence) (case 2).

Final answer

Once we have computed each f_i for each $i < K$, we can add up all of them to get the total number of Special Subsequences with any ending value.

A few values of f_i are listed below:

$f_1 = 1,$

$f_2 = 1 + f_1 = 2,$

$f_3 = 1 + f_1 = 2,$

$f_4 = 1 + f_2 + f_1 = 4,$

$f_5 = 1 + f_1 = 2,$

$f_6 = 1 + f_1 + f_2 + f_3 = 6.$

Thus, we can compute f_i for i upto 22 using a calculator and verify that

$$\sum_{i=1}^{15} f_i = 69$$

$$\sum_{i=1}^{19} f_i = 105$$

$$\sum_{i=1}^{22} f_i = 133$$

ADHISH KANCHARLA is a 11th grade student from Bombay Scottish School in Mumbai. He enjoys solving algorithmic and mathematical problems and has qualified for the American Invitational Math Exam (AIME) and the IOI Training Camp (IOITC) in the 9th and 10th grades. He runs a YouTube channel on competitive programming and has over 3.6K subscribers and 140K views.