Four Problems on Surds

TOYESH PRAKASH SHARMA

In this article we look at four problems and provide solutions to them. In high school we encounter square roots but not roots of higher order. Yet these problems can be solved at that level by observing some patterns and using logic.

Problems

1. Evaluate

$$\left[\frac{6}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7}+1\right)\left(\sqrt[8]{7}+1\right)}+1\right]^{16}$$

2. If $x = 1 + \sqrt[5]{2} + \sqrt[5]{4} + \sqrt[5]{8} + \sqrt[5]{16}$, find the value of

$$\left(1+\frac{1}{x}\right)^{30}$$

3. Show that

(a) $$\frac{1}{6\sqrt[3]{6\sqrt{3}+10}}+\frac{1}{6\sqrt[3]{6\sqrt{3}-10}} = \frac{1}{\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{-10+6\sqrt{3}}}$$

(b) $$\frac{1}{2\sqrt[3]{6\sqrt{3}-10}}-\frac{1}{2\sqrt[3]{6\sqrt{3}+10}} = \frac{1}{\sqrt[3]{10+6\sqrt{3}}-\sqrt[3]{-10+6\sqrt{3}}}$$

4. Calculate

$$\frac{\sqrt[4]{7-4\sqrt{3}}}{\sqrt[4]{7+4\sqrt{3}}}$$

Solutions

1. We have

$$
\begin{aligned}
&\left[\frac{6}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7}+1\right)\left(\sqrt[8]{7}+1\right)}+1\right]^{16}\\
&=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7}+1\right)\left(\sqrt[8]{7}+1\right)\left(\sqrt[8]{7}-1\right)}+1\right]^{16}\\
&=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7}+1\right)\left(\sqrt[8]{7^2}-1\right)}+1\right]^{16}\\
&=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7}+1\right)\left(\sqrt[4]{7}-1\right)}+1\right]^{16}\\
&=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{\left(\sqrt{7}+1\right)\left(\sqrt[4]{7^2}-1\right)}+1\right]^{16}\\
&=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{\left(\sqrt{7}+1\right)\left(\sqrt{7}-1\right)}+1\right]^{16}\\
&=\left[\frac{6\cdot\left(\sqrt[8]{7}-1\right)}{6}+1\right]^{16}\\
&=\left[\left(\sqrt[8]{7}-1\right)+1\right]^{16}\\
&=\left(\sqrt[8]{7}\right)^{16}\\
&=49.
\end{aligned}
$$

2. We have

$$
\begin{aligned}
&x=1+\sqrt[5]{2}+\sqrt[5]{4}+\sqrt[5]{8}+\sqrt[5]{16}\\
&\Rightarrow \sqrt[5]{2}\cdot x=\sqrt[5]{2}+\sqrt[5]{4}+\sqrt[5]{8}+\sqrt[5]{16}+\sqrt[5]{32}\\
&\Longrightarrow \sqrt[5]{2}\cdot x=\left(\sqrt[5]{2}+\sqrt[5]{4}+\sqrt[5]{8}+\sqrt[5]{16}+1\right)+1\\
&\Longrightarrow \sqrt[5]{2}\cdot x=x+1\\
&\Longrightarrow x\cdot\left(\sqrt[5]{2}-1\right)=1\\
&\Longrightarrow \sqrt[5]{2}-1=\frac{1}{x}.
\end{aligned}
$$

Hence

$$\left(1+\frac{1}{x}\right)^{30}=\left(1+\sqrt[5]{2}-1\right)^{30}=2^6=64.$$

3. (a) We have

$$\sqrt[3]{10+6\sqrt{3}}=\sqrt[3]{(1)^3+\left(\sqrt{3}\right)^3+3\sqrt{3}\cdot\left(1+\sqrt{3}\right)}=\sqrt[3]{\left(1+\sqrt{3}\right)^3}=\left(1+\sqrt{3}\right),$$

$$\sqrt[3]{-10+6\sqrt{3}}=\sqrt[3]{(-1)^3+\left(\sqrt{3}\right)^3-3\sqrt{3}\cdot\left(-1+\sqrt{3}\right)}=\sqrt[3]{\left(-1+\sqrt{3}\right)^3}=\left(-1+\sqrt{3}\right)$$

Then $\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{-10+6\sqrt{3}}=2\sqrt{3}$. Now, taking LHS

$$\frac{1}{6\sqrt[3]{6\sqrt{3}+10}}+\frac{1}{6\sqrt[3]{6\sqrt{3}-10}}$$

$$=\frac{1}{6}\left[\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{3}-1}\right]=\frac{1}{6}\left[\frac{\left(\sqrt{3}-1\right)+\left(\sqrt{3}+1\right)}{\left(\sqrt{3}\right)^2-1}\right]$$

$$=\frac{1}{6}\left[\frac{2\sqrt{3}}{2}\right]=\frac{\sqrt{3}}{6},$$

While RHS

$$\frac{1}{\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{-10+6\sqrt{3}}}$$

$$=\frac{1}{\left(\sqrt{3}+1\right)+\left(\sqrt{3}-1\right)}=\frac{1}{2\sqrt{3}}=\frac{1}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{6}.$$

Hence

$$\frac{1}{6\sqrt[3]{6\sqrt{3}+10}}+\frac{1}{6\sqrt[3]{6\sqrt{3}-10}}=\frac{1}{\sqrt[3]{10+6\sqrt{3}}+\sqrt[3]{-10+6\sqrt{3}}}.$$

(b) As in the above problem we have seen $\sqrt[3]{10+6\sqrt{3}}=\left(1+\sqrt{3}\right)$, $\sqrt[3]{-10+6\sqrt{3}}=\left(-1+\sqrt{3}\right)$ and $\sqrt[3]{10+6\sqrt{3}}-\sqrt[3]{-10+6\sqrt{3}}=2$. Then, LHS

$$\frac{1}{2\sqrt[3]{6\sqrt{3}-10}}-\frac{1}{2\sqrt[3]{6\sqrt{3}+10}}$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}\right]=\frac{1}{2}\left[\frac{\left(\sqrt{3}+1\right)-\left(\sqrt{3}-1\right)}{\left(\sqrt{3}\right)^2-1}\right]$$

$$=\frac{1}{2}\left[\frac{2}{2}\right]=\frac{1}{2}$$

While RHS

$$\frac{1}{\sqrt[3]{10+6\sqrt{3}}-\sqrt[3]{-10+6\sqrt{3}}}$$

$$=\frac{1}{\left(\sqrt{3}+1\right)-\left(\sqrt{3}-1\right)}=\frac{1}{2}$$

Hence proved.

4. As

$$\left(1+\sqrt{3}\right)^4=\left(\left(1+\sqrt{3}\right)^2\right)^2=\left((1)^2+\left(\sqrt{3}\right)^2+2\left(1\right)\left(\sqrt{3}\right)\right)^2$$
$$=\left(4+2\sqrt{3}\right)^2=4\cdot\left(2+\sqrt{3}\right)^2=4\cdot\left(4+3+4\sqrt{3}\right)=4\cdot\left(7+4\sqrt{3}\right)$$

then, $\left(7+4\sqrt{3}\right)=\frac{1}{4}\left(1+\sqrt{3}\right)^4\Rightarrow\sqrt[4]{7+4\sqrt{3}}=\frac{1}{\sqrt{2}}\cdot\left(1+\sqrt{3}\right)$ and

$$\left(\sqrt{3}-1\right)^4=\left(\left(\sqrt{3}-1\right)^2\right)^2=\left((1)^2+\left(\sqrt{3}\right)^2-2\left(1\right)\left(\sqrt{3}\right)\right)^2$$
$$=\left(4-2\sqrt{3}\right)^2=4\cdot\left(2-\sqrt{3}\right)^2=4\cdot\left(4+3-4\sqrt{3}\right)=4\cdot\left(7-4\sqrt{3}\right)$$

Then, $\left(7-4\sqrt{3}\right)=\frac{1}{4}\left(\sqrt{3}-1\right)^4\Rightarrow\sqrt[4]{7-4\sqrt{3}}=\frac{1}{\sqrt{2}}\cdot\left(\sqrt{3}-1\right)$.

Now,

$$\frac{\sqrt[4]{7-4\sqrt{3}}}{\sqrt[4]{7+4\sqrt{3}}}$$
$$=\frac{\frac{1}{\sqrt{2}}\cdot(1+\sqrt{3})}{\frac{1}{\sqrt{2}}\cdot(\sqrt{3}-1)}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$
$$=\frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}=\frac{\left(\sqrt{3}+1\right)^2}{\left(\sqrt{3}\right)^2-1}$$
$$=\frac{1+\left(\sqrt{3}\right)^2+2\sqrt{3}}{2}=\frac{4+2\sqrt{3}}{2}=2+\sqrt{3}.$$

TOYESH PRAKASH SHARMA got interested in Science, Mathematics and Literature when he was in 9th standard. He passed 10th and 12th from St. C.F. Andrews School, Agra. When he was in 11th standard, he got interested in doing Research in mathematics and till now many of his works have found place in different journals such as *Mathematical Gazette*, *Crux Mathematicorum*, *Parabola*, *AMJ*, *Pentagon*, *Octagon*, *At Right Angles*, *Fibonacci Quarterly*, *Mathematical Reflections*, etc. Currently he is doing his undergraduation in Physics and Mathematics from Agra College, Agra, India. He may be contacted at toyeshprakash@gmail.com.