On Trisecting an Angle

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Most students come to know in high school that "it is not possible to trisect an angle" using the methods permitted in geometrical constructions; it has become part of mathematical folklore. On hearing that the problem has not been solved by anyone, one immediately feels tempted to tackle it oneself. Who knows, maybe I will hit upon a method that no one has thought of earlier! Whatever the thinking involved, a phenomenon that continues to this day is that of students coming up with various kinds of procedures that they think (or hope!) will work. These procedures are sometimes so complicated that any kind of analysis becomes daunting. (I receive lots of these!)

The geometrical facts of the situation may be stated as follows.

(a) Using only a compass and an unmarked straight edge, it is not possible to exactly trisect an arbitrary angle. (Note that we use the words "unmarked straight edge" rather than "ruler" as the ruler has markings on it.)

(b) It may be possible to trisect some particular angles, by making use of properties that are special to those angles. (For example, one may trisect an angle measuring 90°, using only a compass and an unmarked straight edge.) But such methods do not work in general.

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(c) If we are permitted to use a marked straight edge (i.e., a ruler), angle trisection *is* possible.

(d) It is also possible if we are permitted to use a curve known as the *Archimedean spiral*.

(e) Angle trisection is also possible using paper-folding.

(f) In the above three cases ((c), (d) and (e)), the methods do not qualify as Euclidean.

(g) Since exact trisection of an arbitrary angle is not possible (using only a compass and an unmarked straight edge), we naturally look for approximate methods, by means of which we can obtain an angle that is very close to $\frac{1}{3}$ of any given angle. Numerous methods of this kind are available, which work with varying degrees of accuracy. Some of these will be described later in this article.

The obvious approach, and why it does not work

Consider the simplest possible approach to trisecting an angle. Let $\measuredangle AOB$ be the given angle. By drawing an arc of a circle with centre O to intersect the arms of the angle, we may assume that $OA = OB$ (see Figure 1). We now trisect segment AB. Let the points of trisection be C and D (with C closer to A, and D closer to B). Now suppose that someone claims that $\measuredangle AOB$ has been trisected, i.e., that $\measuredangle AOC = \measuredangle COD = \measuredangle DOB = \frac{1}{3}\measuredangle AOB$. How would we check whether this is so or not?

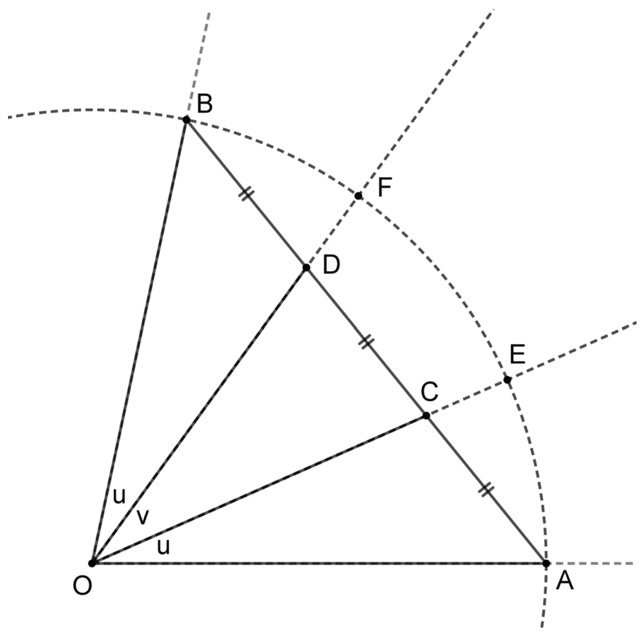

Figure 1

Intuitively, it seems "obvious" that this claim cannot be true. Indeed, it seems obvious that arc EF will be greater in length than arc AE and therefore that $\measuredangle COD > \measuredangle AOC$. But describing something as intuitively obvious does not make it true! We clearly need an argument that is more convincing. (The phrase "this is obvious" has often proved to be quite treacherous; there are numerous instances in the history of mathematics that illustrate this phenomenon.)

To proceed, we first show that $OA > OC$. By design, $OA = OB$, so $\measuredangle OAB = \measuredangle OBA$. Since $\measuredangle ACO$ is an exterior angle to $\triangle OBC$, it follows that $\measuredangle OCA = \measuredangle OBC + \measuredangle BOC$. Therefore $\measuredangle OCA > \measuredangle OBC$. But $\measuredangle OBC = \measuredangle OAC$. Therefore $\measuredangle OCA > \measuredangle OAC$, and so $OA > OC$. (Here we use the known result that in a triangle with two unequal angles, the side opposite the greater angle is longer than the side opposite the smaller angle.)

Using this result, we shall show that $\measuredangle AOC < \measuredangle COD$. For this, we make use of the sine rule (from trigonometry). Let $\measuredangle AOC = u$, and $\measuredangle COD = v$. (Note that $\measuredangle DOB = u$ too.)

Since $\measuredangle ACO$ and $\measuredangle OCD$ are supplementary, $\sin \measuredangle ACO = \sin \measuredangle OCD$. Hence:

$$\frac{\sin u}{\sin v} = \frac{\sin u / \sin \measuredangle ACO}{\sin v / \sin \measuredangle OCD} = \frac{AC/OA}{CD/OD} = \frac{OD}{OA} = \frac{OC}{OA} < 1. \tag{1}$$

Hence $u < v$. Since $2u + v = \measuredangle AOB$, this implies that $u < \frac{1}{3}\measuredangle AOB$ and $v > \frac{1}{3}\measuredangle AOB$.

Unfortunately, from this analysis, we cannot gauge the percentage error in taking u to be $\frac{1}{3}$ of $\measuredangle AOB$. A finer analysis is required for that. We proceed to show how this can be done.

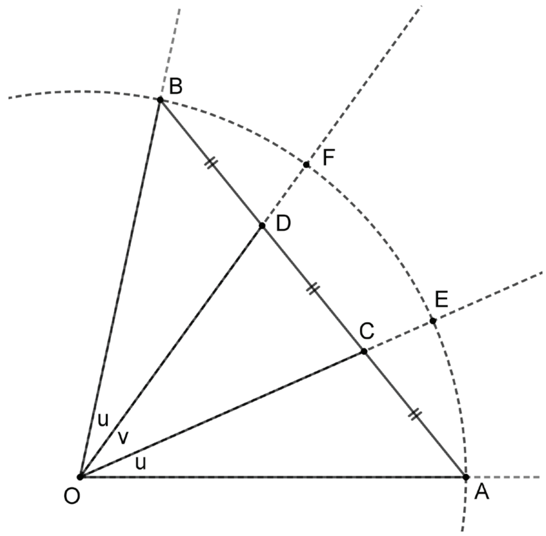

Figure 2

We have redrawn the figure for ease of reading (Figure 2). Let $\measuredangle AOB = t$. Our task now is to express u in terms of t. We now have, from $\triangle AOC$ and $\triangle AOB$,

$$\begin{aligned} \frac{\sin u}{\sin t} &= \left(\frac{(\sin u)/AC}{(\sin t)/AB}\right) \cdot \frac{AC}{AB} \\ &= \left(\frac{(\sin \measuredangle OAC)/OC}{(\sin \measuredangle OAB)/OB}\right) \cdot \frac{1}{3} = \frac{1}{3} \cdot \frac{OB}{OC}. \end{aligned} \tag{2}$$

We now assign coordinates. Take O to be the origin, $O = (0, 0)$, and A to be the 'unit point' on the x-axis, $A = (1, 0)$. Then $B = (\cos t, \sin t)$. Then we have

$$C = \left(\frac{2 + \cos t}{3}, \frac{\sin t}{3}\right). \tag{3}$$

Using this, we determine the length of OC by using the distance formula. We get:

$$OC^2 = \frac{5 + 4\cos t}{9}, \quad \text{after simplification.} \tag{4}$$

Hence:

$$\sin u = \frac{\sin t}{\sqrt{5 + 4\cos t}}. \tag{5}$$

We have thus expressed u in terms of t. The formula allows us to compute u for any given t. We may thus generate the values shown in Table 1.

t	9 °	18 °	27 °	36 °	45 °	54 °	63 °	72 °	81 °	90 °
u	2.997 °	5.98 °	8.92 °	11.82 °	14.64 °	17.36 °	19.95 °	22.39 °	24.61 °	26.56 °

Table 1

As can be seen, u is fairly close to $\frac{1}{3}t$ for small values of t. But the error gets steadily larger as t increases, and for $t = 90°$, the error is close to 11.5% (which is unacceptably large).

Using (5) it is easy to compute the Taylor-Maclaurin series for u in terms of t. We get:

$$u = \frac{t}{3} - \frac{t^3}{81} - \frac{t^5}{972} - \frac{7t^7}{87480} + \cdots. \tag{6}$$

We see directly from (6) that u is always less than $\frac{1}{3}$ of t; that the error is small when t is small; but that the error steadily increases as t increases.

Most such procedures can be analysed in a similar way, though the analysis can get complicated and quite challenging if the number of steps is large. Such is surely the case with the procedure devised by Shri Mahesh Bubna, described elsewhere in this issue. On the other hand, the accuracy level of this procedure is truly astonishing.

Trigonometric analysis of Shri Mahesh Bubna's method

We do not repeat the steps here but plunge straight away into the trigonometric analysis. Let $\measuredangle TBS = t$ and let $f(t) = \measuredangle KBS$. We need to express $f(t)$ in terms of t. Here goes ...:

(1) $B = (0,0)$, $S = (1,0)$, $\measuredangle TBS = t$, $BS = BT = 1$, $T = (\cos t, \sin t)$, $B' = (1 + \cos t, \sin t)$

(2) $BB' = 2\cos\frac{1}{2}t = B'C'$, $C' = (1 + 2\cos\frac{1}{2}t + \cos t, \sin t)$

(3) $Y = \left(1 + \frac{2}{3}\cos\frac{1}{2}t + \cos t, \sin t\right)$, $Z = \left(1 + \frac{4}{3}\cos\frac{1}{2}t + \cos t, \sin t\right)$

(4) From the above we get:

$$\begin{aligned} BY^2 &= 2 + \frac{4}{9}\cos^2\frac{1}{2}t + \frac{4}{3}\cos\frac{1}{2}t\cos t + \frac{4}{3}\cos\frac{1}{2}t + 2\cos t, \\ BZ^2 &= 2 + \frac{16}{9}\cos^2\frac{1}{2}t + \frac{8}{3}\cos\frac{1}{2}t\cos t + \frac{8}{3}\cos\frac{1}{2}t + 2\cos t. \end{aligned}$$

(5) Let $M = (1 + k_1 \cdot \cos t, k_1 \cdot \sin t)$ where $k_1 = \dfrac{SM}{SB'}$. From $BM^2 = BY^2$ we get

$$1 + {k_1}^2 + 2k_1 \cdot \cos t = BY^2, \qquad \therefore\ k_1 = -\cos t + \sqrt{\cos^2 t + BY^2 - 1}.$$

(6) $\measuredangle MBN = \arctan\left(\dfrac{k_1 \cdot \sin t}{1 + k_1 \cdot \cos t}\right)$, $\measuredangle EBN = \dfrac{1}{2}\arctan\left(\dfrac{k_1 \cdot \sin t}{1 + k_1 \cdot \cos t}\right)$, with k_1 as above.

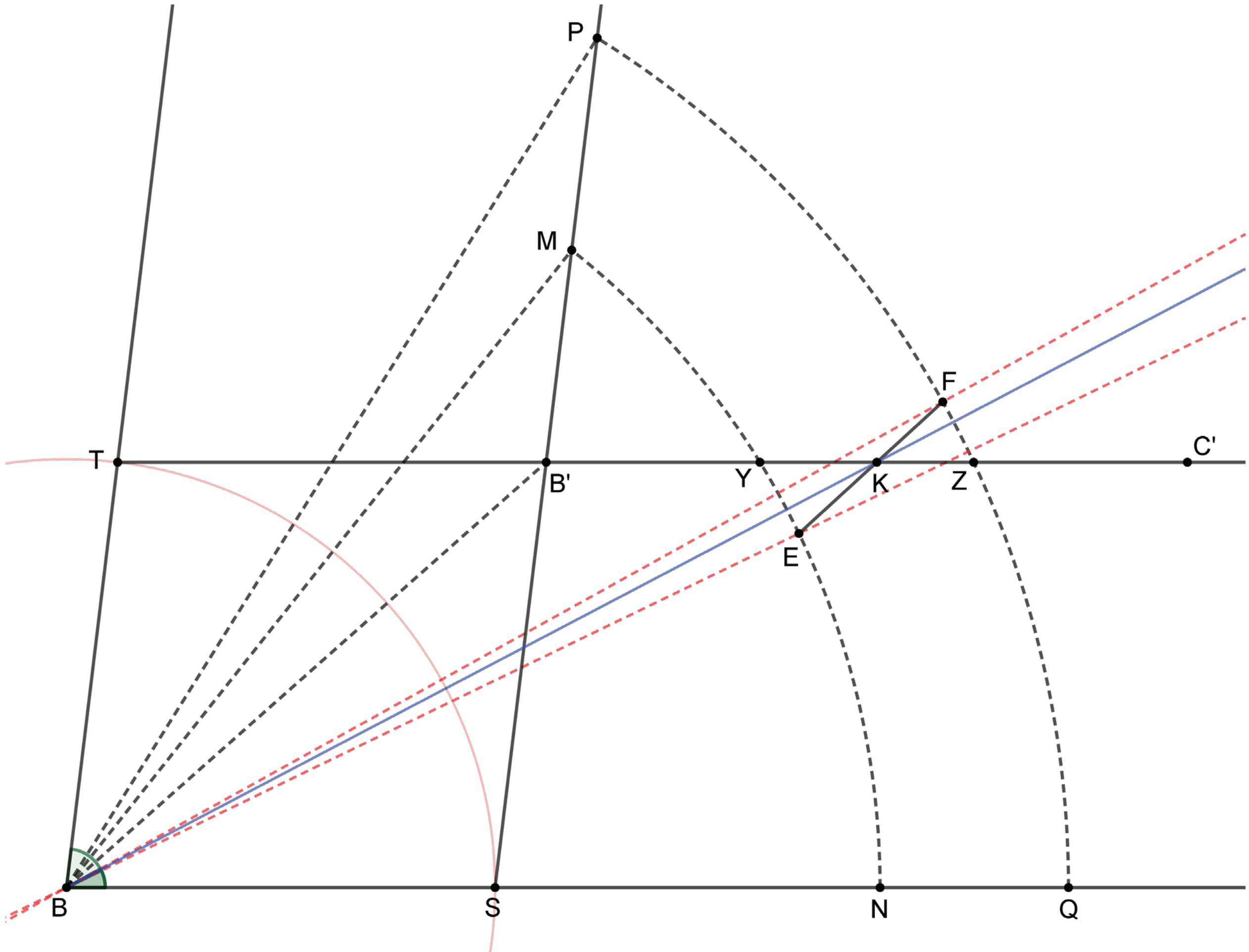

Figure 3

(7) In the same way, let $P = (1 + k_2 \cdot \cos t, k_2 \cdot \sin t)$ where $k_2 = \dfrac{SP}{SB'}$. From $BP^2 = BZ^2$ we get

$$1 + {k_2}^2 + 2k_2 \cdot \cos t = BZ^2, \qquad \therefore\ k_2 = -\cos t + \sqrt{\cos^2 t + BZ^2 - 1}.$$

(8) $\measuredangle PBQ = \arctan\left(\dfrac{k_2 \cdot \sin t}{1 + k_2 \cdot \cos t}\right)$, $\measuredangle FBQ = \dfrac{1}{2}\arctan\left(\dfrac{k_2 \cdot \sin t}{1 + k_2 \cdot \cos t}\right)$, with k_2 as above.

(9) $E = (BY \cdot \cos\measuredangle EBN,\ BY \cdot \sin\measuredangle EBN)$

(10) $F = (BZ \cdot \cos\measuredangle FBQ,\ BZ \cdot \sin\measuredangle FBQ)$

(11) $K = (u, \sin t)$, with u to be determined

(12) $u = BZ \cdot \cos\measuredangle FBQ + \left(\dfrac{BZ \cdot \cos\measuredangle FBQ - BY \cdot \cos\measuredangle EBN}{BZ \cdot \sin\measuredangle FBQ - BY \cdot \sin\measuredangle EBN}\right) \cdot (\sin t - BZ \cdot \sin\measuredangle FBQ)$

(13) $f(t) = \arctan\dfrac{\sin t}{u}$

Using these, we may construct a table of values of $f(t)$, as we did earlier. The result is shown in Table 2.

t	9 °	18 °	27 °	36 °	45 °	54 °	63 °	72 °	81 °	90 °
f(t)	3 °	6 °	8.9997 °	11.999 °	14.998 °	17.997 °	20.995 °	23.993 °	26.989 °	29.985 °

Table 2

The high level of accuracy can easily be seen. When $t = 90°$, the error is just 1 part in 3000. Very impressive!

As earlier we may compute the Taylor-Maclaurin series for $f(t)$. This is difficult to do by hand as the derivation itself is so complicated. We must take recourse to a powerful computer algebra system to do the task. Here is the result:

$$f(t) = \frac{t}{3} - \frac{121t^3}{2073600} - \frac{533179t^5}{159252480000} + \cdots. \tag{7}$$

The smallness of the coefficients of t^3 and t^5 are convincing demonstrations of the high level of accuracy of this procedure.

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