Geometric Proofs of Two Trigonometric Identities

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There are numerous trigonometric identities involving particular angles that have an extremely appealing form. Here is one such:

$$\cos 36^\circ \cdot \cos 72^\circ = \frac{1}{4}. \qquad (1)$$

Before proceeding, let us convince ourselves using trigonometry that this relation is true. To do this we shall derive expressions for $\cos 36^\circ$ and $\cos 72^\circ$. Let $t = 36^\circ$. Then $3t$ and $2t$ are supplementary angles, so they satisfy the relation

$$\cos 3t = -\cos 2t. \qquad (2)$$

Using the well-known triple angle and double angle identities for the cosine function, we get

$$4\cos^3 t - 3\cos t + 2\cos^2 t - 1 = 0 \qquad (\text{for } t = 36^\circ). \qquad (3)$$

Let $x = \cos t$; then the above equality transforms into a cubic equation in x:

$$4x^3 + 2x^2 - 3x - 1 = 0. \qquad (4)$$

We need to solve this equation. Cubic equations do not form part of the standard school syllabus, so one may wonder how to proceed. Fortunately here, the factor theorem shows the way. The substitution $x = -1$ yields 0 on the left side, which means that $x = -1$ is a root of the equation. This in turn means that $x + 1$ is a factor of the cubic polynomial $4x^3 + 2x^2 - 3x - 1$. By division we easily obtain the other factor which is a quadratic:

$$4x^3 + 2x^2 - 3x - 1 = (x + 1) \cdot \left(4x^2 - 2x - 1\right). \qquad (5)$$

Keywords: Trigonometric identity, geometric proof, golden triangle, golden ratio, double angle identity, triple angle identity

The quadratic equation $4x^2 - 2x - 1 = 0$ yields the following two roots:

$$\frac{2 \pm \sqrt{4+16}}{8}, \quad \text{i.e.,} \quad \frac{\sqrt{5}+1}{4} \text{ and } \frac{1-\sqrt{5}}{4}. \tag{6}$$

Hence the roots of the equation $4x^3 + 2x^2 - 3x - 1 = 0$ are -1, $(\sqrt{5}+1)/4$ and $(1-\sqrt{5})/4$. Which of these is the value of $\cos 36^\circ$? Since $0 < \cos 36^\circ < 1$, it must be that

$$\cos 36^\circ = \frac{\sqrt{5}+1}{4}. \tag{7}$$

From this we easily derive an expression for $\cos 72^\circ$, using the double angle identity for cosine, $\cos 2\theta = 2\cos^2\theta - 1$. We obtain:

$$\cos 72^\circ = \frac{\sqrt{5}-1}{4}. \tag{8}$$

Examining (7) and (8), we immediately notice that

$$\cos 36^\circ \cdot \cos 72^\circ = \frac{1}{4}. \tag{9}$$

Observing such a nice-looking result naturally provokes us to ask: Is there a neat geometric argument that will show us why (9) is true, without the use of any trigonometry? Indeed there is, and you can find a very readable version of it at [1]. The proof can be presented as follows.

Roman Andronov's proof of (9) **— a proof (almost) without words.**

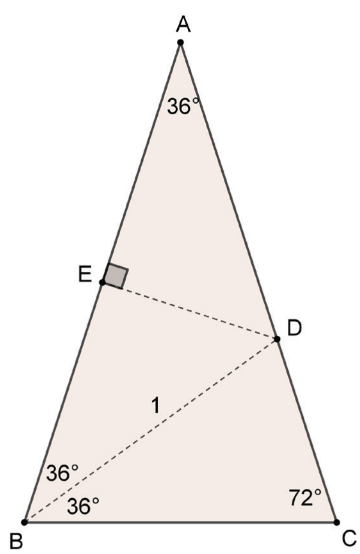

Figure 1

- $\triangle ABC$ has angles 36°, 72°, 72°.
- BD bisects $\measuredangle ABC$, so $\measuredangle ABD = \measuredangle DAB = 36^\circ$, which means that $\triangle DAB$ is isosceles.
- Draw $DE \perp AB$; so DE is also the perpendicular bisector of AB.
- Let $BD = 1$. Then $BC = 1$ too, since $\triangle BCD$ has angles 36°, 72° and 72°.
- From $\triangle DEB$ we get $BE = 1 \cdot \cos 36^\circ$, and so $AB = 2 \cdot \cos 36^\circ$.
- Hence $BC = 2 \cdot AB \cdot \cos 72^\circ$, i.e., $BC = 4 \cdot \cos 36^\circ \cdot \cos 72^\circ$.
- But we also have $BC = 1$.
- Hence $4 \cdot \cos 36^\circ \cdot \cos 72^\circ = 1$, and (9) follows. $\square$

Figure 1 displays a proof of this equality. Given alongside are the steps of the proof. Note that the proof does not require the expressions for $\cos 36^\circ$ and $\cos 72^\circ$.

Another striking relation. If we look again at (7) and (8), we notice a second relation connecting the two quantities which is just as striking as (9):

$$\cos 36^\circ - \cos 72^\circ = \frac{1}{2}. \tag{10}$$

Let us now set ourselves the challenge of proving (10) geometrically (i.e., without explicitly using the expressions for $\cos 36^\circ$ and $\cos 72^\circ$).

The task turns out to be somewhat more difficult than the earlier one. The best that I have been able to come up with is described below. Perhaps some reader can find a shorter approach.

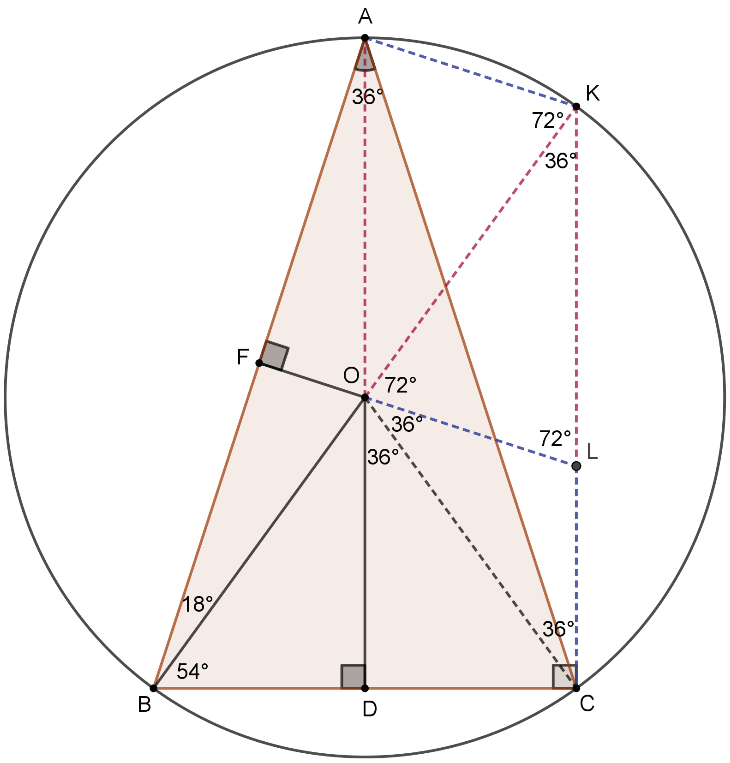

Figure 2

Explanation of the diagram (Figure 2) and steps of the proof.

- $\triangle ABC$ has angles $36^\circ, 72^\circ, 72^\circ$. O is the centre of the circumcircle of $\triangle ABC$. From O drop perpendiculars OD, OF to BC, AB respectively. Then D, F are the midpoints of BC, AB respectively. Draw diameter BK; join AK and KC. Extend FO to meet KC at L. Join OA. Let the radius of the circumcircle be taken as 1 unit.
- From $\measuredangle BAC = 36^\circ$ we get $\measuredangle BOC = 72^\circ$, so $\measuredangle COD = \measuredangle BOD = 36^\circ$. Hence $\measuredangle OBD = 54^\circ$, i.e., $\measuredangle KBC = 54^\circ$, so $\measuredangle BKC = 36^\circ$. Since $BK = 2$, it follows that $KC = 2 \cdot \cos 36^\circ$.
- Also, $\measuredangle OBF = 18^\circ$, so $\measuredangle AKB = 72^\circ$. Hence from $\triangle KAB$, we get $AK = 2 \cdot \cos 72^\circ$.
- Next, observe that $KAOL$ is a parallelogram. (For, $KA \parallel LO$, both being perpendicular to AB; and $AO \parallel KL$, both being perpendicular to BC.) Hence $OL = AK$. So $OL = 2 \cdot \cos 72^\circ$.

- Since $OFBD$ is cyclic, we get $\measuredangle LOD = 72^\circ$, hence $\measuredangle LOC = 36^\circ$. Also, $\measuredangle OCD = 54^\circ$, hence $\measuredangle LCO = 36^\circ$. It follows that $LC = LO$, so $LC = 2 \cdot \cos 72^\circ$.
- Angle-chasing yields $\measuredangle KOL = 72^\circ = \measuredangle KLO$, so $KL = KO$, i.e., $KL = 1$. Hence we have $KC = 1 + 2 \cdot \cos 72^\circ$.
- Earlier we had shown that $KC = 2 \cdot \cos 36^\circ$. It follows that $2 \cdot \cos 36^\circ = 1 + 2 \cdot \cos 72^\circ$.
- Hence $\cos 36^\circ - \cos 72^\circ = 1/2$, as required. □

Remark. A triangle with angles of $36^\circ, 72^\circ, 72^\circ$ is known in the literature as a **golden triangle**. To see why, let $\triangle ABC$ have angles 36°, 72°, 72°. Then we have

$$\frac{AB}{BC} = \frac{1}{2 \cdot \cos 72^\circ} = \frac{2}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{2}. \tag{11}$$

We see that AB/BC is equal to the golden ratio Φ. This explains why it is called the golden triangle.

Postscript. A simpler proof of the second identity

After writing the above proof I came across a vastly simpler proof of (10); it is a true proof-without-words! Here it is.

Proof by Lai Johnny of (10)**.** The proof is from [2]. Curiously, the diagram used is the same as the one used in the proof of (9). We have drawn it afresh here (Figure 3).

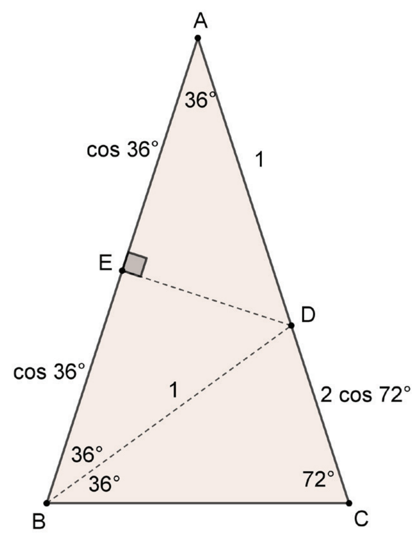

Figure 3. A proof-without-words of the identity $\cos 36^\circ - \cos 72^\circ = 1/2$

From the diagram, it immediately follows that $2 \cdot \cos 36^\circ = 1 + 2 \cdot \cos 72^\circ$, and therefore that

$$\cos 36^\circ - \cos 72^\circ = \frac{1}{2}.$$

A beautiful proof! □

Appendix : the double-angle and triple-angle identities for the cosine function. These are the following identities valid for all θ; they are needed to prove (3):

$$\cos 2\theta = 2\cos^2\theta - 1,$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

References

1. Roman Andronov, "How do I prove that $\cos 36^\circ \cdot \cos 72^\circ = 1/4$ by using the golden triangle $(36^\circ, 72^\circ, 72^\circ)$?" from https://qr.ae/pv0Dgk
2. Lai Johnny, "What is a simple form of $4\sin 36^\circ \cos 72^\circ \sin 108^\circ$?" from https://qr.ae/prR8a0

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