Problem Corner

Interesting Infinite, Recurring Decimals

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The case of the infinite decimal 0.1234567…

We wish to find the fraction corresponding to the infinite decimal $0.1234567\ldots$. Note that we are *not* simply writing the consecutive numbers one after the other to get the infinite decimal. Rather, there is a 'carry-over' effect to the left as we write the successive numbers: once we get to 10, we must 'carry' the 1 to the left, which means it gets added to the 9; and so on. The natural question to ask is: what kind of number will emerge from this construction? Will it be a rational number? Let us look at the question more closely.

The problem expressed more precisely is to find the fraction corresponding to the following infinite decimal:

$$0.1234567\ldots = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{4}{10^4} + \frac{5}{10^5} + \cdots. \tag{1}$$

Offhand, it is not at all obvious whether it will result in a recurring decimal. Let's see where the algebra will lead us. Let a denote the above number. Then we have:

$$\begin{aligned} a &= \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{4}{10^4} + \frac{5}{10^5} + \cdots, \\ \therefore \quad \frac{a}{10} &= \qquad\ \frac{1}{10^2} + \frac{2}{10^3} + \frac{3}{10^4} + \frac{4}{10^5} + \cdots, \\ \therefore \quad \text{(by subtraction)} \quad \frac{9a}{10} &= \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \frac{1}{10^5} + \cdots, \\ \therefore \quad \frac{9a}{10} &= \frac{1}{9} \qquad \left(\text{since } 0.1111\ldots = \frac{1}{9}\right). \end{aligned} \tag{2}$$

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Hence $a = \dfrac{10}{81}$, i.e.,

$$0.1234567\ldots = \frac{10}{81}. \tag{3}$$

We see that the number in question is indeed a rational number, and it is easy now to find the recurring portion corresponding to it. Here is what we get after a straightforward division:

$$\frac{10}{81} = 0.123456790\ 123456790\ \ldots = 0.\overline{123456790}. \tag{4}$$

Curiously, every digit occurs in the decimal expansion — except for 8!

The case of the infinite decimal 0.01020304050607...

We wish to find the fraction corresponding to the infinite decimal $0.01020304050607\ldots$. As earlier, it is assumed that there is a 'carry-over' effect to the left as we write the successive numbers.

The problem expressed more precisely is to find the fraction corresponding to the following infinite decimal:

$$0.01020304050607\ldots = \frac{1}{10^2} + \frac{2}{10^4} + \frac{3}{10^6} + \frac{4}{10^8} + \cdots. \tag{5}$$

As earlier, let's see where the algebra will lead us. Let b denote the above number. Then we have:

$$\begin{aligned}
b &= \frac{1}{10^2} + \frac{2}{10^4} + \frac{3}{10^6} + \frac{4}{10^8} + \frac{5}{10^{10}} + \cdots,\\
\therefore\ \frac{b}{10^2} &= \qquad\ \ \frac{1}{10^4} + \frac{2}{10^6} + \frac{3}{10^8} + \frac{4}{10^{10}} + \cdots,\\
\therefore\ \text{(by subtraction)}\ \frac{99b}{10^2} &= \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \frac{1}{10^8} + \frac{1}{10^{10}} + \cdots,\\
\therefore\ \frac{99b}{10^2} &= \frac{1}{99} \qquad \left(\text{since } 0.01010101\ldots = \frac{1}{99}\right).
\end{aligned} \tag{6}$$

Hence

$$0.01020304050607\ldots = \frac{100}{99^2} = \frac{100}{9801}. \tag{7}$$

Generalization

In the same way, we obtain:

$$0.001002003004005006007\ldots = \frac{1000}{999^2} = \frac{1000}{998001}, \tag{8}$$

$$0.0001000200030004000500060007\ldots = \frac{10000}{9999^2} = \frac{10000}{99980001}, \tag{9}$$

and so on. The pattern should be clear.

Questions for the reader

What happens when we use the sequence of powers of 2 or 3 (or any other number) to produce an infinite decimal? Do we get rational numbers (and therefore recurring decimals)?

$$0.001\ 002\ 004\ 008\ 016\ 032\ 064\ \ldots = ??$$
$$0.0001\ 0003\ 0009\ 0027\ 0081\ 0243\ 0729\ \ldots = ??$$

Note that the 'carry-over' is assumed to happen in these numbers.

What happens when we use the sequence of Fibonacci numbers to produce an infinite decimal? Do we get rational numbers (and therefore recurring decimals)?

$$0.001\ 001\ 002\ 003\ 005\ 008\ 013\ 021\ \ldots = ??$$
$$0.0001\ 0001\ 0002\ 0003\ 0005\ 0008\ 0013\ 0021\ \ldots = ??$$

We leave these questions for the reader to pursue. As can be seen, many such questions can be posed and explored, and they lead to very pretty results.

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