Achu's Primes - A Special Class of Weak Primes

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Twin primes. Pairs of prime numbers that differ by 2 are referred to as *twin prime pairs* or simply *twin primes*. Examples of twin prime pairs are $\{3, 5\}$, $\{5, 7\}$, $\{11, 13\}$, $\{17, 19\}$, $\{29, 31\}$, $\{41, 43\}$, $\{59, 61\}$ and $\{71, 73\}$.

It is conjectured that there are infinitely many twin prime pairs. This is known as the *Twin Prime Conjecture*.

The question of whether there exist infinitely many such pairs has been studied for many centuries but it remains open.

It is clear that every prime number beyond 3 must be of one of the forms $6k \pm 1$. This is so because any number exceeding 3 and of any of the forms $6k$, $6k + 2$, $6k + 3$, $6k + 4$ is composite. This implies that every twin prime pair other than $\{3, 5\}$ is of the form $\{6k - 1, 6k + 1\}$ where k is a positive integer.

From this follows a simple claim:

Lemma 1. *If $p \geq 7$, then $p - 2, p, p + 2$ cannot be an arithmetic progression of primes.*

This is true because if $p \geq 7$ and $\{p - 2, p\}$ is a twin prime pair, then $p + 2$ is necessarily a multiple of 3 and hence composite. □

In the definition below, $[z]$ refers to the greatest integer not exceeding z.

Definition 1. A prime number p is said to be an *Achu's prime* if $\left[\dfrac{p^2+4}{p+2}\right]$ is a prime number.

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For example, 2, 3 and 7 are Achu's primes because

$$\left\lfloor \frac{2^2+4}{2+2} \right\rfloor = 2, \quad \left\lfloor \frac{3^2+4}{3+2} \right\rfloor = 2, \quad \left\lfloor \frac{7^2+4}{7+2} \right\rfloor = 5$$

are all primes. But 5 is not an Achu's prime because

$$\left\lfloor \frac{5^2+4}{5+2} \right\rfloor = 4$$

is not prime. Here are the Achu's primes below 1000:

2,	3,	7,	13,	19,	31,	43,	61,	73,
103,	109,	139,	151,	181,	193,	199,	229,	241,
271,	283,	313,	349,	421,	433,	463,	523,	571,
601,	619,	643,	661,	811,	823,	829,	859,	883.

As of now, the largest known Achu's prime is the following number:

$$2996863034895 \times 2^{1290000} + 1.$$

Lemma 2. *If* $p \geq 7$, *then* $\left\lfloor \frac{p^2+4}{p+2} \right\rfloor = p - 2.$

Note that p does not have to be a prime number for this relation to be true.

Proof of Lemma 2. We only need to show the following: if $p \geq 7$ then

$$p - 2 < \frac{p^2+4}{p+2} < p - 1.$$

The inequality on the left side is equivalent to

$$p^2 - 4 < p^2 + 4,$$

and this is clearly true for all p.

The inequality on the right side is equivalent to

$$p^2 + 4 < p^2 + p - 2,$$

i.e., to $6 < p$, and this is true by supposition, as $p \geq 7$. Hence the statement is true. □

Here is the main result in this paper.

Theorem 1. A prime number $p \geq 7$ is an Achu's prime if and only if $\{p-2, p\}$ is a twin prime pair.

Proof of Theorem 1. Suppose that a prime number $p \geq 7$ is an Achu's prime. Then $\left\lfloor \frac{p^2+4}{p+2} \right\rfloor$ is a prime number (by definition).

By Lemma 2, $\left\lfloor \frac{p^2+4}{p+2} \right\rfloor = p - 2$. So both $p-2$ and p are prime numbers, i.e., $\{p-2, p\}$ is a twin prime pair.

Next, suppose that $\{p-2, p\}$ is a twin prime pair. Then it means that $\left[\dfrac{p^2+4}{p+2}\right]$ is a prime number. But $\left[\dfrac{p^2+4}{p+2}\right] = p-2$. So p is an Achu's prime. □

Definition 2. A prime number is said to be *weak* if it is less than the arithmetic mean of the two prime numbers immediately below and above it.

That is if

$$p_1 = 2,\ p_2,\ p_3,\ \ldots p_{n-1},\ p_n,\ p_{n+1},\ \ldots$$

is the sequence of prime numbers, then p_k is a weak prime if the following inequality is true:

$$p_k < \frac{p_{k-1} + p_{k+1}}{2}.$$

Here are all the weak primes below 1000:

3,	7,	13,	19,	23,	31,	43,	47,	61,
73,	83,	89,	103,	109,	113,	131,	139,	151,
167,	181,	193,	199,	229,	233,	241,	271,	283,
293,	313,	317,	337,	349,	353,	359,	383,	389,
401,	409,	421,	433,	443,	449,	463,	467,	491,
503,	509,	523,	547,	571,	577,	601,	619,	643,
647,	661,	677,	683,	691,	709,	743,	761,	773,
797,	811,	823,	829,	839,	859,	863,	883,	887,
911,	919,	941,	953,	971,	983,	997.		

Theorem 2. If $p \geq 7$ is an Achu's prime, then p is a weak prime.

Proof of Theorem 2. Let $p \geq 7$ be an Achu's prime, and let q and r be the two prime numbers immediately below it and above p.

Using the results already established, we have $q = p-2$.

Since $p \geq 7$ and $p-2$ and p are prime numbers, $p+2$ cannot be a prime number. Hence $r > p+2$.

It follows that $q + r > (p-2) + (p+2)$, i.e., $q + r > 2p$, which tells us that p is a weak prime. This completes the proof of the theorem. □

Concluding remarks.

(1) All Achu's primes can be obtained as the maximum of twin prime pairs.

(2) If there are infinitely many twin prime pairs, then there are infinitely many Achu's primes.

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