Problem Corner

Another Theorem for Congruence of Triangles

KASI RAO JAGATHAPU

We know that any triangle that can be constructed uniquely given three of its six elements gives rise to a congruency relation to other triangles constructed with the same given data.

Based on this core concept, Euclid, in his *Elements-Book 1*, proposed and proved three propositions related to triangle congruency (Proposition-4, Proposition-8 and Proposition-26). These Propositions have appeared as theorems in secondary level Mathematics textbooks as the Side-Angle-Side (SAS), Angle-Side-Angle (ASA) and Side-Side-Side (SSS) congruency theorems respectively. Apart from these, Right Angle-Hypotenuse-Side (RHS) too was introduced in the textbooks.

Are these theorems sufficient to tackle every problem involving the need of establishing congruency between triangles? Let's look at the following problems.

Problem 1

In Figure1, AB and CD are a pair of line segments intersecting each other at K. Segments BC and AD are joined. Prove that if ∠CKB is an obtuse angle, KB = KD, and BC = AD, then AB = CD.

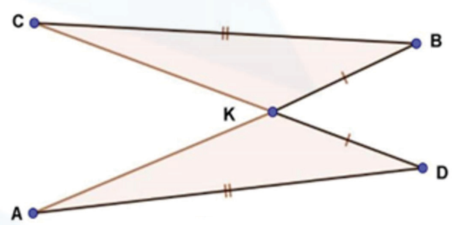

Figure 1

Keywords: Triangles, congruency, sufficient conditions

Here, to prove AB = CD, the need to establish congruency between triangles KAD and KCB is obvious. But none of the theorems we have used so far come to one's aid to prove the required congruency. You can check this for yourself.

Let's examine another problem.

Problem 2

In Figure 2, K is the common centre of two concentric circles. KA is a radius of the inner circle, and AB and AC are a pair of line segments making equal acute angles with AK at A on either side. Prove that AB = AC.

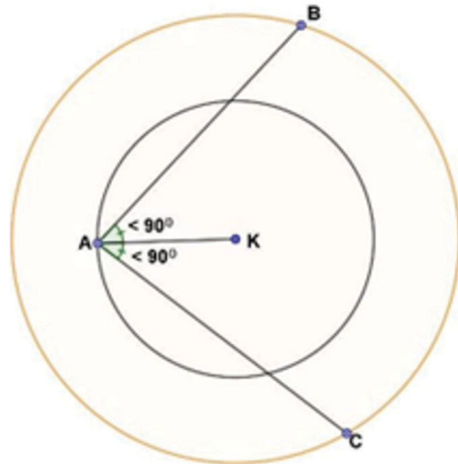

Figure 2

Proof of this result also needs establishment of congruency between the triangles obtained by joining K to B and C. Like Problem 1, this meets the same fate, i.e., no theorem we have in our textbooks will serve the purpose.

Problems such as the above motivate us to introduce and prove additional congruency theorems in secondary level mathematics. To do this, we need to first investigate triangles formed when two sides and a non-included angle are given. The formation of such triangles depends upon the length of the side opposite to the non-included angle relative to the side adjacent to it.

Case-1: When the opposite side is shorter than the adjacent side

Figure 3 shows ΔABC and ΔABC' with BC = BC', with AB and ∠BAC common. It is evident that two different shaped triangles are formed even though there are two pairs of equal sides and a common non-included angle (∠BAC). Here, we note that the side opposite ∠BAC, i.e., BC is *smaller* than the side adjacent to it, i.e., AB. (We consider that adjacent side which is one of the given sides.)

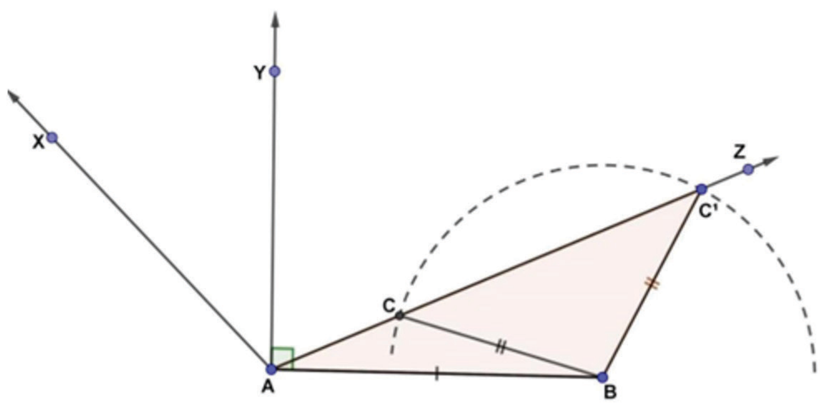

Figure 3

As the triangle so formed is not unique, we conclude that this type of situation for a pair of triangles does not lead to their congruency.

Case-2: When the opposite side is equal to the adjacent side

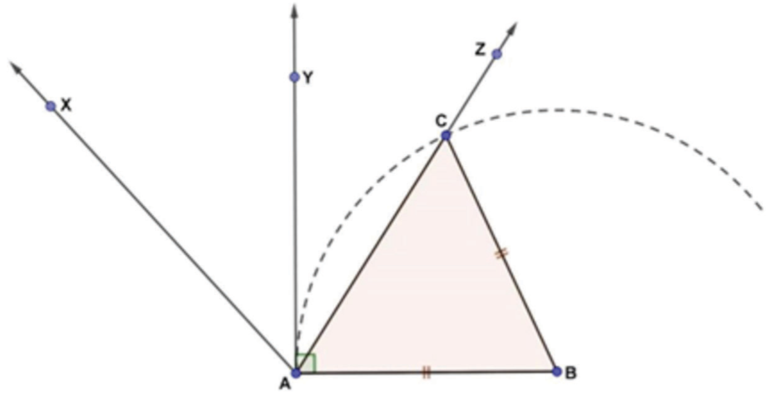

Figure 4

Figure 4 shows that a unique isosceles triangle forms when the side BC opposite to the non-included angle ∠BAC is equal to the side AB adjacent to it. So, this criterion can be considered for establishment of congruency with another triangle of same nature. By SAS congruency criterion, congruency can be established with same measurements. Hence this situation doesn't demand any new proposition.

Case-3: When the opposite side is greater than adjacent side

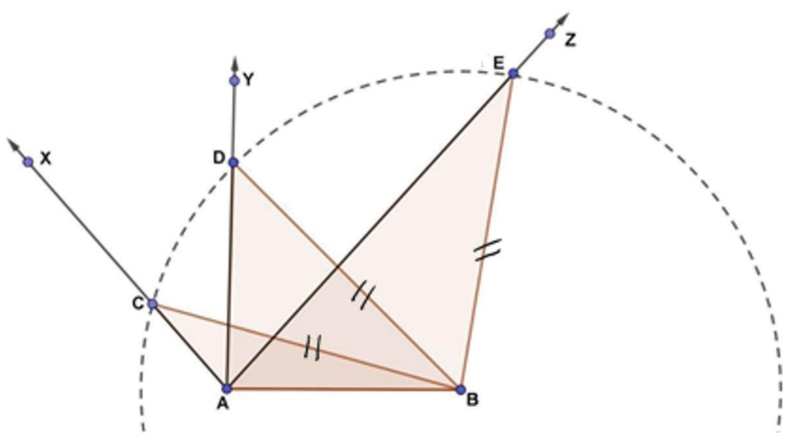

Figure 5

Figure 5 clearly shows that three different triangles – obtuse angled, right angled, and acute-angled can form, and their formation is unique. Hence these kinds of triangles do have the relation of congruency with another triangle of similar nature.

The congruency theorem pertaining to right-angled triangles which is tagged as RHS find place in our textbooks. But the other two results (obtuse-angled and acute-angled cases, with the condition about opposite side to non-included angle) do not appear in our secondary level mathematics curriculum.

The two problems cited at the beginning of this article demand the introduction of theorems of congruency relating to obtuse-angled triangles (can be tagged as 'OSS') and acute-angled triangles (can be tagged as AaSS), just like the RHS congruency theorem.

Obtuse angle - Side - Side Congruency Theorem (OSS)

"If there exists a correspondence between the vertices of two obtuse angled triangles in such a way that two sides and the non-included obtuse angle of one triangle are respectively equal to two corresponding sides and non-included obtuse angle of the other, then the two triangles are congruent."

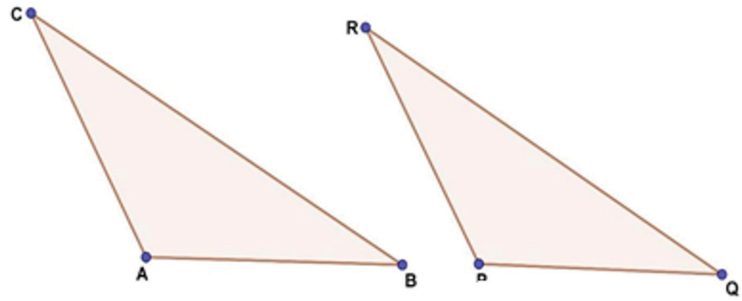

Figure 6

Hypothesis: ΔABC and ΔPQR are two obtuse angled triangles in which

1. Obtuse $\angle BAC$ = Obtuse $\angle QPR$
2. $AB = PQ$
3. $BC = QR$

Conclusion: $\Delta ABC \cong \Delta PQR$

PROOF: Assume ΔABC is not congruent to ΔPQR

Because $\angle BAC = \angle QPR$, it is possible to shift the ΔPQR so that P coincides with A, PR in the direction of AC. See Figure 7.

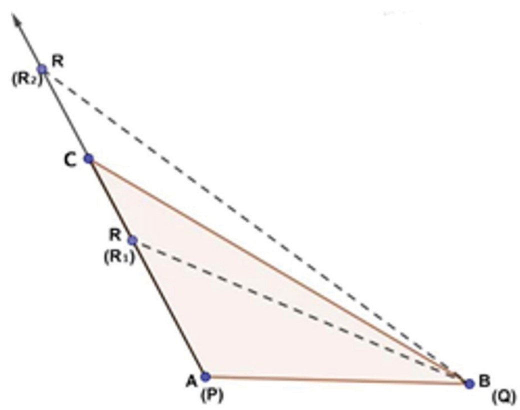

Figure 7

Also, as $AB = PQ$, Q coincides with B.

As per assumption, triangle ABC is not congruent to triangle PQR, so R does not coincide with C.

Hence vertex R should lie either below C or above C on the line AC. Let the two possibilities be indicated by points R_1 and R_2 respectively.

Consider ΔABR_1, in which $\angle AR_1B$ is an acute angle (since $\angle A$ is obtuse).

In ΔBR_1C, $\angle BR_1C$ is an obtuse angle (linear pair of $\angle AR_1B$), so BC is greater than BR_1.

Now BR_1 is nothing but QR, which implies that BC is greater than QR.

But by hypothesis, $BC = QR$, which is a contradiction.

So, our assumption that ΔABC is not congruent to ΔPQR is incorrect.

Hence $\Delta ABC \cong \Delta PQR$.

The same logic can be applied to ΔCBR_2 also.

Therefore $\Delta ABC \cong \Delta PQR$.

Note: Right angle - side (Hypotenuse) - Side (RHS) Theorem too can be proved by contradiction using the same logic.

Acute angle - Side - Side (AaSS) Theorem

In Figure 8, ΔABC and ΔPQR are two acute angled triangles in which $\angle A = \angle P$ (both are acute) and $AB = PQ$, $BC = QR$. If AC and PR are the longest sides of Δs ABC & PQR respectively, then the two triangles are congruent.

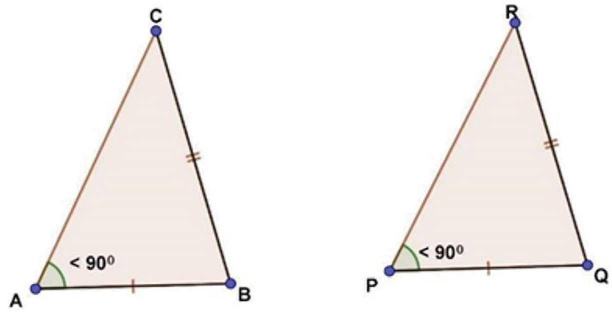

Figure 8

This also can be proved by applying the same logic of proof by contradiction as done in the case of OSS theorem.

The criteria we have covered for congruency in triangles – OSS, RHS and AaSS – may be stated in theorem form as follows:

If two unequal sides of a triangle are equal to two sides of another triangle, and the angles opposite the longer of the two sides are equal, then the two triangles are congruent.

If two equal sides of a triangle are equal to two equal sides of another triangle, and the angles opposite two corresponding equal sides are equal, then the triangles are congruent.

KASI RAO JAGATHAPU is a retired high school headmaster now residing in the town of Kavali in Andhra Pradesh. He taught mathematics for 35 years. He greatly enjoys reading books and finding solutions to problems and puzzles. He received the 'Best Teacher' award in two different districts. He may be contacted at kasiraoj@gmail.com.