On the Circumcentres of Isosceles Trapezia

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All isosceles trapezia are cyclic, that is, they all have circumcircles. The circumcentre could be inside, on or outside the figure. If it is on the figure, it must coincide with the midpoint of the longer of the parallel sides (generally considered the 'base'). In any case, the circumcentre must lie on the line of symmetry of the figure.

The question we now ask is: Given the lengths of the sides of an isosceles trapezium, is it possible to determine if its circumcentre lies inside, on or outside the figure (without actually constructing it)?

Let us name the longer of the parallel sides as a, the shorter parallel side as b, and each of the equal sides as c. Let us also define $m = a/2$, $n = b/2$ and h as the height of the figure (distance between the parallel sides). The height is related to the sides of the figure by the following equation: $h^2 = c^2 - (a - b)^2/4$ (see Figure 1, where CE is perpendicular to AB).

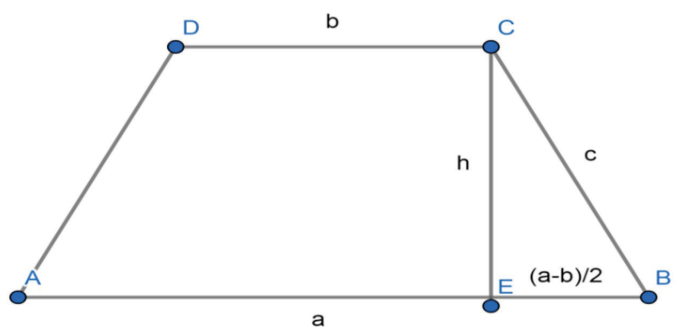


Figure 1

Keywords: Isosceles trapezia, cyclic quadrilaterals, circum-centre, coordinate geometry, pure geometry

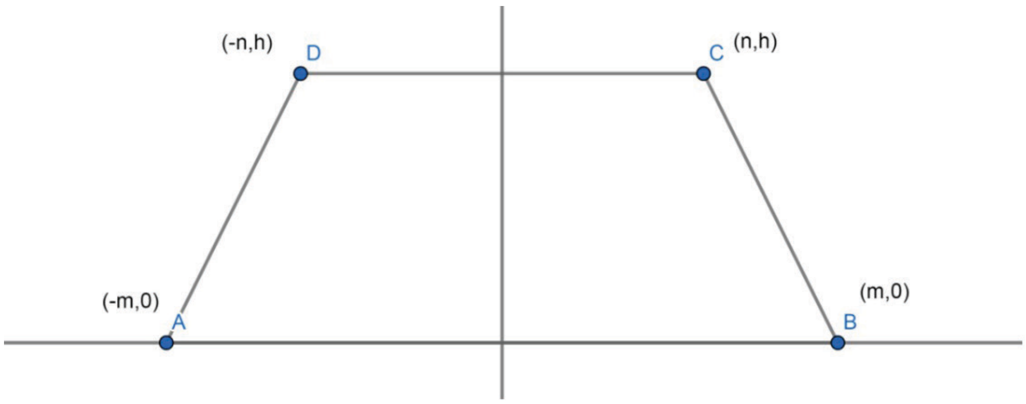


Figure 2

We first take an approach based on coordinate geometry to answer the above question. Let the isosceles trapezium ABCD be placed on the Cartesian plane with the base AB on the X-axis, and with the Y-axis as the line of symmetry (see Figure 2). The coordinates of the vertices are as given in the figure.

The circumcentre of the figure must be at the point of intersection of the perpendicular bisector of BC and the line of symmetry (the Y-axis). The equation of the perpendicular bisector of BC is

$$(x-m)^2 + y^2 = (x-n)^2 + (y-h)^2.$$

As the circumcentre lies on the Y-axis, we can substitute $x = 0$, to find its y-coordinate. Then the above equation simplifies to

$$m^2 + y^2 = n^2 + y^2 + h^2 - 2hy,$$

$$\text{or } y = (n^2 - m^2 + h^2)/2h.$$

If the circumcentre lies on side AB, $y = 0$, i.e.,

$$n^2 - m^2 + h^2 = 0,$$

$$\text{or } h^2 = m^2 - n^2, \text{ or } h^2 = (a^2 - b^2)/4.$$

Substituting for h^2, we have

$c^2 - (a-b)^2/4 = (a^2 - b^2)/4$, which simplifies to $2c^2 = a(a-b)$, the condition for the circumcentre to lie on the base.

If the circumcentre lies inside the isosceles trapezium, $y > 0$, so

$$h^2 > m^2 - n^2, \text{ or } 2c^2 > a(a-b).$$

If the circumcentre lies outside the isosceles trapezium, $y < 0$, so

$$h^2 < m^2 - n^2, \text{ or } 2c^2 < a(a-b).$$

We now obtain the same relations by a pure geometry approach. The length of the diagonal d of an isosceles trapezium can be obtained from the side lengths by the relation $d^2 = ab + c^2$, which follows from Ptolemy's theorem. (This theorem states that in cyclic quadrilateral ABCD, AB.CD + BC.AD = AC.BD.)

Consider ΔABD in isosceles trapezium ABCD (Figure 3). If $\angle$ADB is a right angle, then the circumcentre of ABCD is the midpoint of AB. In that case, $a^2 = c^2 + d^2$, so $a^2 = 2c^2 + ab$, or $2c^2 = a(a-b)$.

If $\angle$ADB is acute, then ABCD occupies a major segment of its circumcircle, with circumcentre inside the figure. In that case,

$$a^2 < c^2 + d^2, \text{ so } a^2 < 2c^2 + ab, \text{ or } 2c^2 > a(a-b).$$

If $\angle$ADB is obtuse, then ABCD is confined to a minor segment of its circumcircle, with circumcentre outside the figure.

In that case, $a^2 > c^2 + d^2$, so $a^2 > 2c^2 + ab$, or $2c^2 < a(a-b)$.

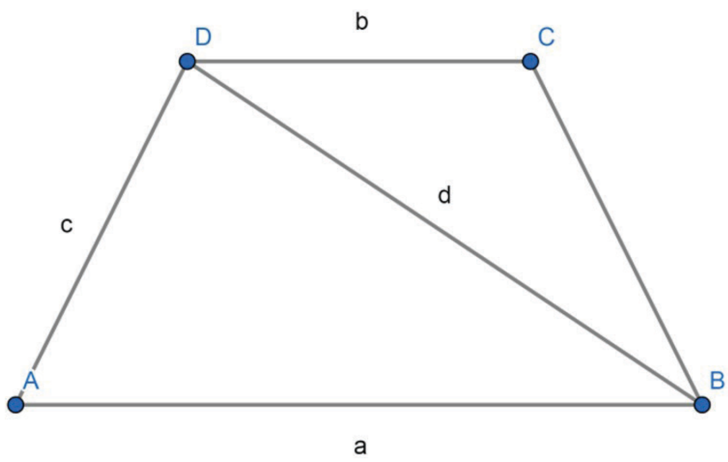
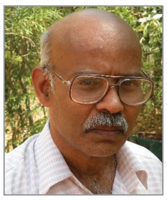


Figure 3



A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.