ClassRoom

To Factor or Not to Factor

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Factorisation of algebraic expressions is an integral part of school algebra curriculum. Most of the expressions one deals with involve at most two variables and their degrees are either 2 or 3. The process of finding factors relies heavily on the ability to make astute observations and clever reductions to some standard forms such as $u^2 - v^2$ or $u^3 \pm v^3$. Rarely does one encounter expressions of degree 4 or higher with the exception of $x^4 + x^2y^2 + y^4$ which can be reduced to the $u^2 - v^2$ form by writing it as

$$(x^4 + 2x^2y^2 + y^4) - x^2y^2 = (x^2 + y^2)^2 - (xy)^2.$$

An interesting question which arises in the problem of factorising an expression of degree 4 or more is: "When do we terminate the factorization process?" That is, if we succeed in obtaining a linear factor of the given expression, should we be satisfied and stop at that point? Or should we look for more factors? Here is an example where such a question comes up very naturally.

A degree 4 expression arises in geometry as

$$16\Delta^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$$

where Δ is the area of a triangle with sides of lengths a, b and c. This expression can also be reduced to a product of four linear factors involving a, b and c with integer coefficients, by repeated application of $u^2 - v^2$ to suitably adjusted algebraic expressions.

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But do we actually check after obtaining a factorisation of an expression that indeed the expressions thus derived cannot be reduced further? Let me make my point with the help of an example. A standard exercise is to factorise

$$a^3 + b^3 + c^3 - 3abc$$

as

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

Students generally stop here because it matches the answer given at the end of the book. But why should one stop here? How does one know that the expression

$$a^2 + b^2 + c^2 - ab - bc - ca$$

cannot be written as a product of two linear factors involving a, b and c with integer coefficients? The only way to know if it can or can't be done is by answering the following question:

Do there exist integers p, q, r, u, v and w such that

$$a^2 + b^2 + c^2 - ab - bc - ca = (pa + qb + rc)(ua + vb + wc)$$

is an identity in a, b and c?

By equating the coefficients one readily observes that

$$\begin{aligned} & pu = qv = rw = 1 \\ \text{and} \quad & pv + uq = qw + vr = ru + pw = -1. \end{aligned}$$

Since each of p, q, r, u, v and w must be ± 1, each of $pv + uq$, $qw + vr$ and $ru + pw$ is even, and hence cannot be equal to -1. This contradiction establishes the impossibility of finding integers p, q, r, u, v and w such that

$$a^2 + b^2 + c^2 - ab - bc - ca = (pa + qb + rc)(ua + vb + wc)$$

is an identity in a, b and c, and justifies the decision to proceed no further after obtaining

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

Are the students taught this way in schools? This is a point to ponder.

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