ClassRoom

Derivation of Formulae for Solving Some Common Arithmetic Problems

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At present, multiple-choice questions (MCQ) and very short answer type questions (VSA) are common features of almost every examination. While they can be solved from first principles, this is often time consuming. In this discussion, some general formulae to solve three common types of arithmetic problems are derived. It is hoped that rather than memorizing these formulae, students will learn to develop and use their own short-cuts as part of their preparation for examinations.

A Problem of Simple and Compound Interest

Problem 1. Find the difference between the simple and compound interest, compounded annually, on ₹10000 for two years when the rate of interest is 5%.

In order to derive a formula for the difference between simple and compound interest, compounded annually, we generalise this problem.

Find the difference between the simple and compound interest, compounded annually, on ₹p, for n years when the rate of interest is r%.

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Solution: Compound interest on ₹p for n years at r% rate of interest $= ₹p\left[\left(1+\frac{r}{100}\right)^n - 1\right]$

Simple interest on ₹p for n years at r % rate of interest $= ₹\,\frac{npr}{100}$

Therefore, difference between compound and simple interest

$$= ₹p\left[\left(1+\frac{r}{100}\right)^n - 1 - \frac{nr}{100}\right]$$

$$= ₹p\left[1+\frac{nr}{100}+\frac{n(n-1)}{2!}\left(\frac{r}{100}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{r}{100}\right)^3\right.$$

$$\left. + \ldots\ldots\ldots + \left(\frac{r}{100}\right)^n - 1 - \frac{nr}{100}\right] \text{ [using binomial expansion]}$$

$$= ₹p\left[\frac{n(n-1)}{2!}\left(\frac{r}{100}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{r}{100}\right)^3+\ldots+\left(\frac{r}{100}\right)^n\right]$$

This is the general formula for finding the difference between annually compounded compound interest and simple interest for n years when the principal is ₹p and rate of interest is r%. The above formula clearly shows that, as expected, for one year ($n = 1$) there is no difference between compound and simple interest. The difference starts from the second year ($n = 2$) and it gradually increases with increasing n, as more and more terms of the difference formula come into play.

Thus, for two years ($n = 2$) the difference $= ₹p\left(\frac{r}{100}\right)^2$

For three years ($n = 3$) the difference $= ₹p\left[3\left(\frac{r}{100}\right)^2+\left(\frac{r}{100}\right)^3\right]$ and so on.

In Problem 1, $p = 10000$, $r = 5$ and $n = 2$, so the difference between the interests = ₹25.

A Problem Related to Mixtures

Problem 2. A drum contains 20 litres of milk. 5 litres of milk are taken out from it and replaced by an equal amount of water. With the resulting mixture, this process is repeated 3 more times, i.e., 4 times in all. What will be the ratio of milk to water in the final mixture?

Here also we first generalize the above problem.

A pot contains x litres of milk. From that container, 'a' litres of milk are taken out and replaced by an equal amount of water. With the resulting mixture, this process is repeated $(n - 1)$ times, i.e., n times in all, each time removing 'a' litres of the mixture and replenishing it by the same amount of water.

Solution: After the first operation, the amount of milk in the mixture is $(x - a)$ litres and the amount of water is a litres. Now, $x - a = x\left(1-\frac{a}{x}\right)$. We observe the following pattern:

Op. No.	Milk taken out (litres)	Milk left (litres)	Simplified form for milk left
1	a	$x - a$	$x\left(1-\frac{a}{x}\right)$
2	$\frac{a(x-a)}{x}$	$\left[(x-a)-\frac{a(x-a)}{x}\right]$	$x\left(1-\frac{a}{x}\right)^2$
3	$a\left(\frac{x-a}{x}\right)^2$	$\frac{(x-a)^2}{x}-\frac{a(x-a)^2}{x^2}$	$x\left(1-\frac{a}{x}\right)^3$

Using symmetry, let us suppose that after m operations, the amount of milk left over is

$$x\left(1-\frac{a}{x}\right)^{\mathrm{m}} \text{ litres, i.e., } \frac{(x-a)^m}{x^{m-1}} \text{ litres.}$$

Then, during the $(m+1)^{\text{th}}$ operation, the amount of milk taken out $= a\dfrac{(x-a)^m}{x^m}$ litres.

So, the amount of milk left after $(m+1)$ operations $= \left[\dfrac{(x-a)^m}{x^{m-1}} - a\dfrac{(x-a)^m}{x^m}\right]$ litres

$= x\left(1-\dfrac{a}{x}\right)^{m+1}$ litres.

Thus, we see that if the formula for the remaining amount of milk is valid for $n = m$, then it is true for $n = m+1$ also. But it has already been shown that the result is true for n $= 1, 2, 3$. Then, by the Principle of Mathematical Induction, the result is true for all positive integral values of n.

Therefore, after n operations the amount of milk left in the mixture $= x\left(1-\dfrac{a}{x}\right)^n$ litres.

So, the amount of water in the final mixture $= x\left[1-\left(1-\dfrac{a}{x}\right)^n\right]$ litres. Then, after n operations, the ratio of milk to water in the mixture $= \left(1-\dfrac{a}{x}\right)^n : \left[1-\left(1-\dfrac{a}{x}\right)^n\right]$

For Problem 2, $x = 20$, $a = 5$, $n = 4$. So, after 4 operations, the amount of milk in the mixture

$= 20\left(1-\dfrac{5}{20}\right)^4$ litres $= \dfrac{405}{64}$ litres.

Amount of water after 4 operations $= \left(20-\dfrac{405}{64}\right) = \dfrac{875}{64}$ litres.

So, ratio of milk to water in the final mixture $= 81 : 175$.

A Problem Related to Percentages

Problem 3. If the price of a watch is increased initially by 10% and then by 6%, then what will be the ultimate percentage increase in the price of the watch?

Here again we first generalize the problem for deriving a formula for direct calculation.

If the price of a commodity is increased in two stages by x% and y% respectively, then what will be the final percentage increase in the price of the commodity?

Solution: Suppose the initial price of the commodity was p.

Then, after x% increase, its price $= p + \dfrac{px}{100} = p\left(1+\dfrac{x}{100}\right)$

After y% increase its final price $= p\left(1+\dfrac{x}{100}\right) + \dfrac{py}{100}\left(1+\dfrac{x}{100}\right)$

$= p\left(1+\dfrac{x}{100}\right)\left(1+\dfrac{y}{100}\right)$

Therefore, final increase in price $= p\left(1+\dfrac{x}{100}\right)\left(1+\dfrac{y}{100}\right) - p$

$= \dfrac{p}{100}\left(x+y+\dfrac{xy}{100}\right)$

So, percentage increase in the price of the commodity $= \frac{p}{100}\left(x + y + \frac{xy}{100}\right)\frac{100}{p}$

$= \mathrm{x} + \mathrm{y} + \frac{xy}{100}$

Therefore, if the price of a commodity is increased initially by x% and subsequently by y% then the actual percentage increase of the price $= \left(x + y + \frac{xy}{100}\right)$

Now, for Problem 3, $x = 10$ and $y = 6$.

Therefore, percentage increase in the price of the watch $= \left(10 + 6 + \frac{60}{100}\right) = 16\frac{3}{5}$

Corollary 1. If the price is initially increased by x% and then decreased by y%, then in the above formula y will be replaced by $-y$.

So, percentage change in price $= \left(x - y - \frac{xy}{100}\right)$.

Corollary 2. Similarly, if the price is decreased by x% and then increased by y% , then x will be replaced by $-x$.

So, percentage change in the price $= \left(y - x - \frac{xy}{100}\right)$.

Corollary 3. If the price is decreased in two stages, viz., first by x% and then by y%, then both x and y will have to be replaced by $-x$ and $-y$.

Therefore, percentage change in the price $= \left(\frac{xy}{100} - x - y\right)$.

Note. The above formulae can be used for similar types of problems such as change in the area of a rectangle due to change in its length and breadth, hike and reduction of salary of a person, etc.

Final Remarks

In the present article direct formulae for solving three different types of arithmetic problems have been derived. In Problem 1, the compound interest is considered as compounded annually. What will be the corresponding formulae if the compound interest is compounded half-yearly, quarterly, thrice a year, etc.? Another point is if the formulae derived above are represented graphically, then what will be the nature of those graphs? These questions are left as exercises for interested readers.

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