A Conversation with Ian about Squares and Cubes

IAN KOMENAKA
& JAMES METZ

Ian: “What is the difference of the squares of two consecutive positive integers?” (He knew the answer, but wanted to challenge me. He said he had discovered a shortcut.)

James: “I’ve made a few examples and it looks like you add the two numbers.” (I had not observed that before.) “How did you discover this?”

Ian: “When I was building some things, I needed to know some areas, so I just made a multiplication table for the integers from 17 through 28.” (See Photograph 1.)

	17	18	19	20	21	22	23	24	25	26	27	28
17	**289**	306	323	340	357	374	391	408	425	442	459	476
18	306	**324**	342	360	378	396	414	432	450	468	486	504
19	323	342	**361**	380	399	418	437	456	475	494	513	532
20	340	360	380	**400**	420	440	460	480	500	520	540	560
21	357	378	399	420	**441**	462	483	504	525	546	567	588
22	374	396	418	440	462	**484**	506	528	550	572	594	616
23	391	414	437	460	483	506	**529**	552	575	598	621	644
24	408	432	456	480	504	528	552	**576**	600	624	648	672
25	425	450	475	500	525	550	575	600	**625**	650	675	700
26	442	468	494	520	546	572	598	624	650	**676**	702	728
27	459	486	513	540	567	594	621	648	675	702	**729**	756
28	476	504	532	560	588	616	644	672	700	728	756	**784**

Table 1. Ian’s multiplication table

James: “What did you notice in the table that made you think about this? Do you recall which numbers you first observed?”

Keywords: Pattern, numbers, observation, exploration, connection

Ian: "I looked at 20 and 21 and saw that their sum was the same as 441 – 400 in the table. Then I tried all the others along the diagonal and the same thing happened. The difference of the squares of two consecutive positive integers was always the sum of the two integers. Then I tried some big number like 46 and 47 and some small numbers and it was still true. I tried all the integers from 1 to 50 and they all worked."

James: "You must have had a pretty strong feeling that it is always true."

Ian: "I think so."

James: "Have you tried to prove it?"

Ian: "Not yet."

James: "Why don't you try to prove it now using what you know from algebra?"

Ian wrote, "x is the smaller number; $x + 1$ is the larger number." He then simplified $(x + 1)^2 - x^2$ to $2x + 1$, which is the same as $x + x + 1$, the sum of the two numbers, and he said, "It works!"

James: "Very good. Before we continue, let's do a little geometry to show $7^2 - 6^2$. How can you represent 7^2?"

Ian: "I can draw a square that is 7 by 7."

James: "Good. Please do that."

James: "Now that you have that, outline a 6 by 6 square. If we take away the 6 by 6 square, you see that we have an 'L' shape that is the difference of the squares. How many unit squares are there?" (See Figure 1.)

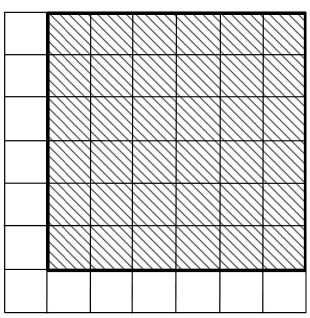

Figure 1. The geometry of $7^2 - 6^2$

Ian: "13."

James: "How can you count them?"

Ian: "7 + 6, or 6 + 6 + 1 which is 2(6) + 1. So, instead of 6, if I use x it will still be true that the difference is $1 + x + x = 2x + 1$." (He could *see* $2x + 1$.)

James: "Great. We will come back to this later. Now I ask you, 'What is the difference of the squares of two consecutive positive even integers?' Try $8^2 - 6^2$ for example."

Ian: "I get 28."

James: "Right. What do 8 and 6 have to do with 28? Remember what you did before."

Ian: "I add them together and then I need to multiply by 2."'

James: "What do you think the rule might be?"

Ian: "2 times the sum. Probably 2 because the difference between them is 2."

James: "Prove it."

Ian simplified $(x + 2)^2 - x^2$ to $4x + 4$ or $2(2x + 2)$ proving that he was correct. He then showed it using the squares on graph paper and generalized from the picture.

James: "You are doing good. What if we have two consecutive positive odd integers?"

Ian: "It will be the same as the evens because they are also separated by 2."

James: "Very good! I have one more question before we move on to something else. What if the difference between the two positive integers is d, where d is a positive integer?"

Ian quickly wrote $(x + d)^2 - x^2$ and announced excitedly, "It is $d(2x + d)$, like I suspected!" He checked a few examples to be satisfied that it worked.

James: "Let's draw a picture for the case $d = 4$, like $7^2 - 3^2$. (See Figure 2.) Can you please describe the "L" shape here?"

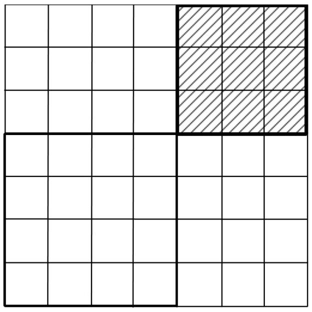

Figure 2. The geometry of $7^2 - 3^2$

Ian looked at the picture and said, "One leg has 3 rows (in general x) with 4 (in general d) squares in each and so does the other one, so there are 2(3)(4) squares (in general $2xd$), plus the corner has 4^2 squares (in general d^2)." Ian generalized easily because he could *see* the rule in the picture.

James: "We have done everything with the difference of the squares of two positive integers, so what shall we do next?"

After thinking about squares for a while, Ian decided to look at the difference of the cubes of two consecutive numbers. He wrote $4^3 - 3^3$ but could not see anything, so he said, "Let me just prove it." He simplified $(x + 1)^3 - x^3$ to $3x^2 + 3x + 1$. (No wonder he could not see any pattern!)

James: "How can we model this with geometry?"

Ian: "We can use cubes."

Using sugar cubes, we looked at $4^3 - 3^3$. After he removed the 27 cubes that made up 3^3, he had 3 yellow "walls", each 3 by 3, 3 white "columns", each 3 by 1 and 1 cube in the corner (not visible in the figure), so he had $3x^2$ (the 3 walls), $3x$ (the 3 columns), and 1 (cube in the corner). (See Figure 3.) The formula made sense.

We decided to call it a day knowing that if we wanted to, we could find the formula for the difference of the cubes of any 2 positive integers.

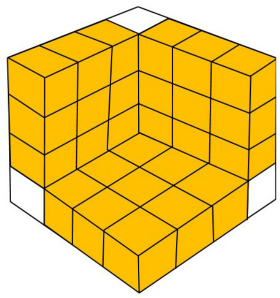

Figure 3. The geometry of $4^3 - 3^3$

Note. At the time of this conversation, Ian was a student in grade 9 in Arizona. Ian's grandmother was the officemate of James for 10 years. The preceding is a transcript of a FaceTime® call. Perhaps the conversation will inspire teachers to delve deeper when students ask questions.

IAN KOMENAKA is currently a student at Arizona College Prep high school in Chandler, Arizona, U.S.A. He has been intrigued by math seemingly for his entire life. He noticed the pattern studied while making a multiplication table for numbers 17-30. He uses math and pattern recognition in his other interests which include chess and football. In the future, he intends to use his interest to pursue a career in statistics. He can be reached at iangk022@gmail.com.

JAMES METZ is a retired mathematics instructor. He volunteers for Teachers Across Borders Southern Africa for one month each year and enjoys doing mathematics. He can be reached at metz@hawaii.edu.