Features

A (S)Trip with a Twist

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The Mobius strip has been an intriguing mathematical object since its discovery in the nineteenth century by August Ferdinand Mobius. Quite interestingly, it also appeared in the Draft National Education Policy 2019 to symbolise the "perpetual, developing, and live nature of knowledge"—an ongoing pursuit of knowledge creation and dissemination. The Mobius Strip has been an object full of wonder and surprise that even magicians couldn't resist, and they exploited its properties to play some mysterious tricks on their audience. The exploration described here also began with one such trick, conducted during a session with pre-service elementary teachers during the first year of their course. The tasks related to exploring Mobius strips provided meaningful opportunities for engaging prospective teachers in mathematical processes, including searching for patterns, visualizing, predicting, verifying, formulating generalizations, and posing new problems. It is being assumed that an active, collaborative, and intellectual engagement with such carefully designed tasks would provide pre-service teachers with a wide range of resources to draw upon and enliven their classroom learning environments.

The Magic Trick

Following a magician's trick from Gardner's book *Mathematics, Magic, and Mystery* [1], three long paper bands were presented, and the pre-service teachers were called upon to cut each of these bands along the middle lengthwise. Initially, it seemed like a routine task that would result in two identical pieces. Cutting the first band resulted in two paper bands, as expected. However, they were in for a

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big surprise as cutting the second strip resulted in a single paper band, longer than the original one. And, from the third one, two interlocked paper bands were obtained. An object like a paper strip that seemed so familiar at first is now seen to be behaving in many strange ways. Students were now curious to look for an explanation for the same, setting the stage for an exhilarating ride ahead. Many of them could guess by now that not all the bands were "normal" paper bands they were familiar with. There was a twist (rather, many twists!) to the story.

Let's twist!

Students were guided to prepare the three paper bands whose properties had been exploited to trick the audience. The first band was a simple loop with no twist (resembling a cylinder or a ring). The second was the Mobius Strip, formed by twisting one end 180 degrees with respect to the other before pasting the two ends together, called a 'half-twist'. The third loop was formed by twisting one end 360 degrees relative to the other before pasting the ends. All three bands are shown in Figure 1.

Figure 1. The yellow band is a simple loop, the pink one is the Mobius strip, and the green one is the strip with two half-twists.

Naturally, this led to a question: What difference does adding a half-twist make to the paper strip? This led to an opportunity to explore the properties of the strips. Students were asked to draw a line along each of the strips until they returned to the starting point. On a simple paper band (with no twists), the line ran along the outer surface only. It is a two-sided surface, with an inside and an outside. However, on a Mobius strip, the line ran along both the surfaces. It was with the introduction of this one-sided surface that many exciting mathematical adventures began to unfold.

A mathematician confided
That a Mobius strip is one-sided.
You'll get quite a laugh
If you cut it in half,
For it stays in one piece when divided.
~ Anonymous

Students were amused to discover the difference that a half twist could make to the properties of a paper strip.

A wave of excitement ran through the class as students went on adding more half twists to their paper strips and cutting them along the middle. In the process, they were encouraged to predict the results before cutting these strips and to keep records of their work so as to leave written traces of their ways of thinking and reasoning about the task. Figure 2 shows the resulting strips after cutting the strips in Figure 1 along the mid-line. Pictures 3 and 4 show the strips obtained after cutting the strips with three and four half twists, respectively.

Figure 2

Figure 3. A long knotted strip.

Figure 4. One strip looped twice around the other.

Much to their amazement, the students could now see that the tabulated observations seemed to be unfolding a pattern. Table 1 offers a glimpse of the observations recorded by one of the groups.

Number of half twists in original band	Number of sides	When cut along the middle line
0	2	Two separate but identical bands Length same as original band Width is half the original
1	1	A single band with 4 half twists Length twice the original Width half the original
2	2	Two interlocking bands with 2 half twists in each (i.e., identical with the original) Length same as original band Width half the original
3	1	A single knotted band with 8 half twists Length twice the original Width half the original
4	2	Two interlocking bands with 4 half twists in each Length same as original band Width half the original

Table 1

The generalised expressions based on a strip with n half twists are:

If n is even,

The bands are all two-sided. If this band is cut along the midline between the edges, we obtain two interlocking bands, each of which has n half twists (identical with the original band). One band is looped $n/2$ times around the other. Both bands are, however, narrower.

If n is odd,

The bands are all Mobius-like, i.e., one-sided surfaces. Cutting along the midline in this case, we obtain a single long band with $2n + 2$ half twists. The resulting strips are, therefore, two-sided and not Mobius-like. Also, for $n > 1$, the resulting band is knotted.

The exploration initiated them into imagining lots of cutting experiments and, at the same time, searching for patterns and making a number of generalizations.

According to Lockhart (2009) [2],

> *A good problem is something you don't know how to solve. That's what makes it a good puzzle and a good opportunity. A good problem does not just sit there in isolation but serves as a springboard for other interesting questions.*

The above exploration led to some new questions, like what would happen if the cuts were not made along the middle of the twisted strips, but at some other distance from the edge? Will that change the resulting strip, and how?

Lo and behold!

Students began investigating with the Mobius strip first. In Figure 5, the yellow strip is obtained after cutting a Mobius strip along 1/3rd the distance from the edge, while the pink strip is obtained after cutting along 1/4th the distance.

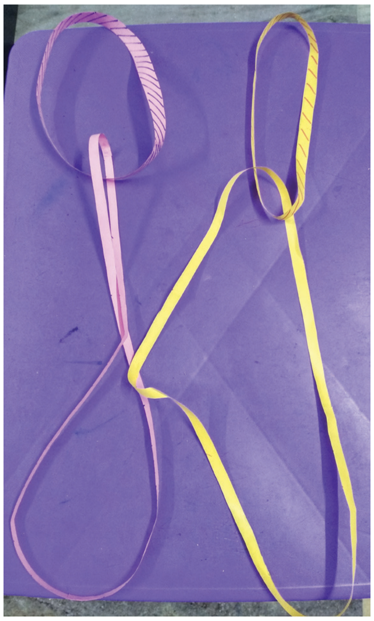

Figure 5

The following observations were made and tabulated as shown in Table 2:

Distance from the Edge	When cut is made
½	A single band with 4 half twists Length twice the original Width half the original
⅓	Two linked bands One Mobius band identical to the original Length same as original; Width one-third of the original One band with 4 half twists Length twice the original; Width one-third of the original
¼	Two linked bands One Mobius band identical to the original Length same as original; Width half of the original One band with 4 half twists Length twice the original; Width one-fourth of the original
⅕	Two linked bands One Mobius band identical to the original Length same as original; Width three-fifths of the original One band with 4 half twists Length twice the original; Width one-fifth of the original

Table 2

Some of the groups could arrive at generalisations based on the observations made. It was expressed as follows:

Cutting a Mobius strip along a $1/m$ distance from the edge resulted in the following linked bands:

The Mobius band obtained has its length same as the original, while the width varied at each stage and is $1 - 2/m$ of the original.

Another band obtained is two-sided and has four half twists in it. Its length is twice the original, while the width is $1/m$ of the original. This band, thus obtained, is identical with (though narrower) the one obtained after cutting the original band along the midline.

A careful look at the cuts being made brought forth an interesting observation. That is, if we start cutting the band at one-third distance from the edge, we traverse the band twice before coming back to the starting point. This explains how the one-third part at the middle of the original band forms a narrower Mobius band identical with the original, along with another band with length twice the original, and width one-third of the original. This is in contrast to cutting the strip along the middle when the strip is traversed only once before coming to the starting point.

A question that arose was: Will it hold for other Mobius-like bands (i.e., bands with an odd number of half twists) too?

This was found to be true for all the bands with an odd number of half twists. We just need to vary the number of half twists in the second band which was again identical (though narrower) with the resulting band when the original band was cut down the middle lengthwise.

The exploration was now being extended to tabulate observations when bands with "n" half twists are cut at $1/m$ distance from the edge.

Number of half twists in original band	Cut along the Distance from the Edge		
	½	⅓	¼
0			
1			
2			
3			

Table 3

The following generalisations were reached:

When the cut is made along $1/m$ distance from the edge:

If the number of half twists is even, 2 interlocking bands, each having n half twists and the same length as the original band, are obtained. The widths of the bands are $1/n$ and $1 - 1/n$.

If the number of half twists is odd, 2 linked bands are obtained. One is Mobius-like, identical with the original, with its width as $1 - 2/n$; another band has a length twice the original, a width of $1/n$, and has $2n + 2$ half twists.

It was quite fascinating how experiments in cutting some twisted strips led to formulating generalizations. Many shifts were witnessed in the students' ways of approaching and engaging with the tasks: from random guessing to being analytical; from simply solving problems to posing new problems; from random note-taking to systematic organising and synthesising information; and towards appreciating beauty in mathematics.

A meaningful engagement in mathematical processes will not only support teachers in emphasising these processes in their own practice but will also prove essential in the sense that these processes cut across different content domains and at all levels. Open-ended tasks allow for multiple entry points and offer multiple pathways for further exploration, charting unfamiliar territories. There is no end to what can be investigated. And so the (s)trip continues...

References

[1] Gardner, M. (1956). *Mathematics, Magic and Mystery*. NY: Dover Publications.

[2] Lockhart, P. (2009). *A Mathematician's Lament.* NY: Bellevue Literary Press.

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