Problem Corner

Products of Sums of All Co-Primes from 1 till Any Given Number

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We are familiar with the following problem: *given a positive integer n, find the sum $g(n)$ of all the positive integers from 1 to n that are coprime to n.* The formula for the required sum is simple:

$$g(n) = \begin{cases} 1, & \text{for } n = 1, \\ \frac{1}{2} \times n \times \Phi(n), & \text{for } n > 1, \end{cases} \tag{1}$$

where $\Phi(n)$ is the Euler totient function (it counts the positive integers from 1 to n that are coprime to n). The reason for formula (1) should be clear: for $n > 1$, if $1 \le a < n$ and we have $\gcd(a, n) = 1$, then we also have $1 \le n - a < n$ and $\gcd(n - a, n) = 1$. This means that the positive integers between 1 and n that are coprime to n can be paired with one another such that the sum of the numbers in each pair is n. Hence the stated result.

In this article, I explore the following problem:

Problem. Find the product $f(n)$ of all such sums up to a given positive integer n.

In other words, we want the formula for the following in terms of the prime factorization of n.

$$f(n) := \prod_{k=1}^{n} g(k). \tag{2}$$

Keywords: Prime number, coprime, greatest common divisor (gcd), Euler's totient function

Example. Let $n=10$. The sums of all the co-primes for $1,2,3,4,5,6,7,8,9,10$ are (respectively) 1, 1, $1+2$, $1+3$, $1+2+3+4$, $1+5$, $1+2+3+4+5+6$, $1+3+5+7$, $1+2+4+5+7+8$, $1+3+7+9$, i.e., $1,1,3,4,10,6,21,16,27,20$, respectively. Hence:

$$\begin{aligned} f(10) &= \text{product of the sums of the co-primes from 1 to 10} \\ &= 1\times 1\times 3\times 4\times 10\times 6\times 21\times 16\times 27\times 20 \\ &= 2^{10}\times 3^{6}\times 5^{2}\times 7. \end{aligned}$$

It is easy to obtain $f(n)$ for small values of n. However, it takes too much time for large numbers. We seek a usable formula for $f(n)$.

Theorem. For $n\geq 9$ we have:

$$f(n)=\frac{1}{2^{n-1}}\times (n!)^2\times\left(\frac{\prod\limits_{k=2}^{k=n}\prod\limits_{p\mid k}(p-1)}{\prod\limits_{k=2}^{k=n}\prod\limits_{p\mid k}p}\right). \tag{3}$$

Proof of theorem. We have $\Phi(1)=1$, and for $n>1$,

$$\Phi(n)=n\prod_{p\mid n}\left(1-\frac{1}{p}\right). \tag{4}$$

Hence, from (1), for $n>1$:

$$g(n)=\frac{1}{2}\times n\times\Phi(n)=\frac{1}{2}\times n^2\times\prod_{p\mid n}\left(1-\frac{1}{p}\right). \tag{5}$$

Hence:

$$\begin{aligned} f(n) &= \prod_{k=1}^{k=n} g(k) \\ &= \prod_{k=2}^{k=n}\frac{1}{2}\times k^2\times\prod_{p\mid k}\left(1-\frac{1}{p}\right) \\ &= \frac{1}{2^{n-1}}\times (n!)^2\times\prod_{k=2}^{k=n}\prod_{p\mid k}\left(1-\frac{1}{p}\right) \\ &= \frac{1}{2^{n-1}}\times (n!)^2\times\frac{\prod\limits_{k=2}^{k=n}\prod\limits_{p\mid k}(p-1)}{\prod\limits_{k=2}^{k=n}\prod\limits_{p\mid k}p}. \end{aligned} \tag{6}$$

Using (6), we may compute the values of f. See Table 1 for a list of some values of the function.

n	f(n)
1	1
2	1
3	3
4	$2^2 \times 3$
5	$2^3 \times 3 \times 5$
6	$2^4 \times 3 \times 5$
7	$2^4 \times 3^3 \times 5 \times 7$
8	$2^8 \times 3^3 \times 5 \times 7$
9	$2^8 \times 3^6 \times 5 \times 7$
10	$2^{10} \times 3^6 \times 5^2 \times 7$
11	$2^{10} \times 3^6 \times 5^3 \times 7 \times 11$
12	$2^{13} \times 3^7 \times 5^3 \times 7 \times 11$
13	$2^{14} \times 3^8 \times 5^3 \times 7 \times 11 \times 13$
14	$2^{15} \times 3^9 \times 5^3 \times 7^2 \times 11 \times 13$
15	$2^{17} \times 3^{10} \times 5^4 \times 7^2 \times 11 \times 13$

Table 1. Values of $f(n)$ for $1 \leq n \leq 15$, expressed in terms of the primes $\leq n$

To find an explicit formula for f, we need to work out the power to which each prime $p \leq n$ occurs in the above quantity. We look at each term in (6) separately.

Let $p_1 = 2$, $p_2 = 3$, p_3, . . ., p_s be the primes not exceeding n. (So there are s prime not exceeding n.)

- For $q \in \{p_1, p_2, \ldots, p_s\}$, the power to which q divides $n!$ is

$$\left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{q^2} \right\rfloor + \left\lfloor \frac{n}{q^3} \right\rfloor + \cdots.$$

  This follows from Legendre's formula [1] which gives an expression for the exponent of the largest power of a prime q that divides $n!$.

  Hence the power to which q divides $(n!)^2$ is

$$2\left\lfloor \frac{n}{q} \right\rfloor + 2\left\lfloor \frac{n}{q^2} \right\rfloor + 2\left\lfloor \frac{n}{q^3} \right\rfloor + \cdots. \tag{7}$$

- Next, the power to which q divides $\prod_{k=1}^{k=n} \prod_{p|k} p$ is $\left\lfloor \frac{n}{q} \right\rfloor$.

  To see why, we simply count the number of occurrences of q in the expression $\prod_{k=1}^{k=n} \prod_{p|k} p$.

- Hence, the power to which q divides $\frac{(n!)^2}{\prod_{k=1}^{k=n} \prod_{p|k} p}$ is

$$\left\lfloor \frac{n}{q} \right\rfloor + 2\left\lfloor \frac{n}{q^2} \right\rfloor + 2\left\lfloor \frac{n}{q^3} \right\rfloor + 2\left\lfloor \frac{n}{q^4} \right\rfloor + \cdots. \tag{8}$$

- Now we must compute the power to which an arbitrary prime number q divides the quantity

$$\prod_{k=1}^{k=n} \prod_{p|k} (p-1). \tag{9}$$

  Further investigation will have to be undertaken to give an explicit formula for (9). The formula may be somewhat complex.

References

1. Wikipedia, "Legendre's formula" from https://en.wikipedia.org/wiki/Legendre%27s_formula

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